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A SOFTWARE TOOL FOR QUALITY FUNCTION DEPLOYMENT INCORPORATING THE QUANTIFICATION METHOD OF TYPE 3

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Abstract

There might be over thirty software products regarding Quality Function Deployment (QFD). Most of them have been developed based on the book titled "Quality Function Deployment" edited by Akao. In the book, the procedure for making a quality table consists of a demanded quality deployment table and a quality element deployment table, which both are composed by using the KJ method or the affinity diagram method. This sometimes brings about a question whether the hierarchy of each table is reasonable or not.

Shindo (1993), and Xiong and Shindo (1995) reported the significance of structuralization of a two-dimensional table and proposed a specific method applying the quantification method of type 3 (QM3). The structuralization enables one to simultaneously arrange or sort the items of both deployment tables. Furthermore, during the structuralization processes, we may be able to detect overlooked items and examine the reasonableness of the hierarchy of grouping.

Recently we have developed a software tool for helping application of this method to a quality table. The tool is designed to apply QFD flexibly and consistently to various products and services together with the structuralization using QM3. This paper describes the procedure of the structuralization using QM3, and how the software tool works with it.

Introduction

As is well known, Professor Yoji Akao has proposed QFD as a quality assurance methodology in a new product development in late 1960s, and the concept of quality deployment has been implemented in Japanese industries since early 1970s. Since Akao introduced QFD to the United States in 1983, QFD has become the major force in the total quality effort in the United States. Nowadays, many people in several countries pay attention to QFD and have begun to consider implementing it according to their new product development policies.

Akao (1990) pointed out that QFD, when appropriately applied, can reduce the development time by one-half to one-third. However, QFD includes a lot of procedures to do, which may increase the development time conversely if we do not organize the work efficiently.

Today, there might be over thirty software products supporting QFD processes. Most of them have implemented the basic procedures proposed in the book titled "Quality Function Deployment" edited by Akao (1990). However, their major efforts have been devoted to supporting making a quality table and quality planning. This means that those software products do not necessarily support the QFD totally.

The procedures for making a quality table recommended in the book include the following steps:

(1) First, make a demanded quality deployment table by the KJ method or the affinity diagram method.

(2) Next, make a quality element deployment table in the same manner.

(3) Then, combine both tables as a quality table and evaluate the degrees of correspondence between items of each table.

The procedures consist of two grouping processes. One is for a demanded quality deployment table; and the other is for a quality element deployment table. In the course of grouping, we sometimes face the question whether the hierarchy of each table is reasonable or not. If different criteria are applied to each grouping, symbols (or any non-zero numbers) representing degrees of correspondence will appear in a sparsely scattered way in the whole table.

This paper briefly introduces the concept of structuralizing a two-dimensional table, which Shindo (1993), and Xiong and Shindo (1995) have reported. They employed the Quantification Method of Type 3 (QM3) as a specific means for structuralization, which wielded a new procedure for composing a quality table without the KJ method or the affinity diagram method. The software tool proposed here also includes this structuralization method, so one can efficiently make a well-structured quality table.

Background

The Design Policy

Although there have been many software tools developed to support QFD processes, most of them pay attention to supporting the processes for making a quality table or quality planning. This means that the software tools do not necessarily support all the QFD processes. Then, we considered the various tables used in QFD processes and abstracted them in order to treat in the same manner.

The first requirements of specification of this software tool are described as below:

(1) Apply the QM3 to simultaneously structuralizing a two-dimensional table for composing a reasonable hierarchy. At this time grouping or ungrouping is up to users.

(2) Reuse the existing table.

(3) Input methods can be chosen dependently on how the data (which compose the two-dimensional table) is accumulated.

(4) Support the quality planning processes.

(5) Easily calculate the weight of every item within the entire two-dimensional table.

(6) Save and reuse data flexibly.

Except for the numbers (2) and (3), all of the requirements of specification have been implemented in this software tool. The number (1) "structuralization using QM3" of the requirement specification is one of the distinguished features offered and implemented in this software tool.

Structuralization Using QM3

It can be said that there are following two important problems in composing a quality table by using conventional procedures with KJ method (or the affinity diagram method).

(1) KJ method is used twice for a demanded quality deployment table and a quality element deployment table. In such a case, our experiences indicate that we have an almost diagonalized quality table if the same criteria are used in grouping. However, it is not assured that criteria used are equal to each other. As a result, we sometimes have an ill-structured quality table.

(2) We sometimes find out an item which has a low level of abstraction located in the higher level of hierarchy, and conversely. However, it is very difficult, in general, for us to detect such missgrouping. This results from the vagueness or ambiguity existing in our expressions.

Above two problems always annoy us in composing a quality table. From such a viewpoint, Shindo (1993) proposed a new method for composing a quality table using the QM3, in which degrees of correspondence between items are effectively utilized.

QM3 (see appendix) is one of methods for analyzing an inside-structure which may exist in a two-dimensional table. QM3 results in an eigen value problem, which produces eigen values and scores (quantified values) correspondent to the eigen vectors. These scores can be used for sorting or rearranging the row and/or column items of the table. Moreover, we can visualize the structure of these items by plotting the points, whose locations are determined by pairs of scores, in a scattered diagram. The degree of affinity between two items is defined as a distance between two points representing those items. Items with the high degree of affinity are plotted near close by and vice versa. The positional relations between items on the scattered diagram can also be used to decide the grouping of the items. This situation is relevant to the idea called system's near decomposition proposed by Simon.

Herbert A. Simon (1990) stated that the complicatedness has hierarchy and it comes from the limit of human's simultaneous understandability. Therefore, one should accomplish not arranging but discovering a structure from the complicatedness. At least such recognition is significant as has been carried out in designing a solid body. Moreover, Simon also proposed an idea of system's being nearly decomposable which seems to be very closely related to discovering a structure of the systems or products. Unfortunately, Simon did not give the definite procedure to achieve the system's near decomposition. That is why we study composing a two-dimensional table by applying QM3.

The structuralization applying QM3 enables one to simultaneously arrange or sort row and/or column items of a two-dimensional table. In the structuralization processes we may be able to detect overlooked items and examine the reasonableness of the hierarchy of grouping. This process is carried out interactively and repeatedly until a convinced table is obtained.

Software Tool Specification

Data structure and functions

One of the design policies of the software tool is to support the QFD totally. This means that the tool must compose not only a quality table but also other tables used in QFD process. To make this process possible, first we abstracted all of the factors consisted in QFD, such as quality, element, cost, etc., and then we considered all possible the two-dimensional tables by combining a pair of these factors. This also enables one to see a certain table from different two viewpoints. For example, one can see the quality table from the viewpoints of both the demanded quality and the quality element. Here, we picked up 9 factors such as demanded quality (Q), quality element (E), cost (C), function (F), technology (T), reliability (R), subsystem (S), process (P) and mechanism (M), and all two-dimensional tables combining a pair of different two factors can be composed. In addition to one-dimensional and two-dimensional tables, we prepare one special two-dimensional table for quality planning, which combines the demanded quality deployment table and quality planning. The idea of modeling the items and tables within QFD is shown in Figure 1.

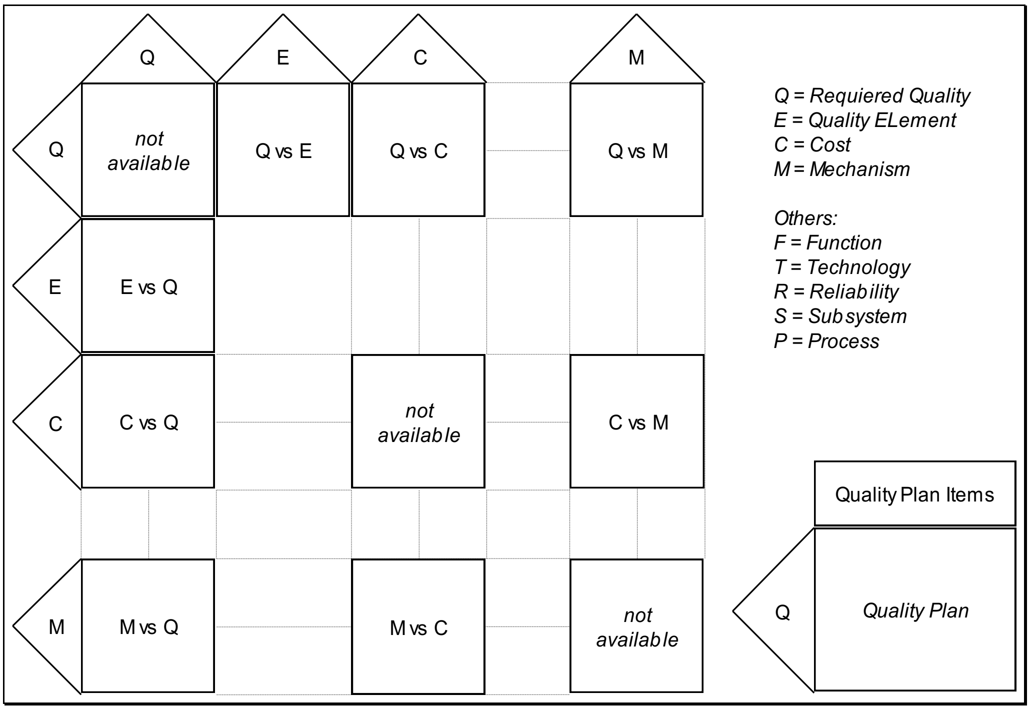

Figure 1 Model of Items and Tables within QFD

The principal features of function provided in the software tool are shown below:

(1) Data entry of item on each factors including its hierarchical level up to 5 level.

(2) Data entry of the degree of correspondence of a table can be entered.

(3) Weight calculation and conversion are automatically supported.

(4) Excluding the quality plan table, the structuralization using QM3 can be applied to each two-dimensional table.

(5) Entered data can be saved and can be reused for editing.

Implementation of Structuralization Using QM3

Formerly, we had to use some independent application programs to carry out the QFD processes including structuralization using QM3. Since the procedure of structuralization should be carried out interactively and repeatedly until a convinced table is obtained, it is not an efficient way to perform the procedure by using non-integrated environment of software tool. This implementation of structuralization using QM3, which characterized this software, enables one to carry out the QFD processes more efficiently.

The structuralization procedure can be applied to each of two-dimensional tables excluding the quality-planning table. In the course of grouping items in a structuralization process, various kinds of subgroups will appear. Some seems to be well structured, while others are not yet. In such a case, we must reapply the QM3 to the latter. Then, the software tool is designed to help one repeatedly apply the QM3 to an arbitrarily specified region within the two-dimensional table without influencing the rest of the entries of the table. It saves time for separating the table and making a new two-dimensional for those separated tables.

The results of QM3 analysis is displayed in a 'dialog-box', which shows the eigen-values and scores correspond to each item in row and column of applied region in the two-dimensional table. Then, the sort or rearrangement of the items of the applied region can be performed based on the appropriate scores. This will instantly change the structure of the applied region.

As the result of the sort or rearrangement of the two-dimensional table, if it is necessary to rearrange the structure of the items regarding the row or column of the table, the changes could be performed in the appropriate window.

User Interface

To perform better efficiency in editing the items of a table, it is realized to edit the data simultaneously. For example, we want to enter the items of quality element while referring to the items of demanded quality. Or, we want to compose the quality table while doing the quality planning procedure. Taking these needs into consideration, we employed the technique called "multi document interface (MDI)" for the software tool. However, unlike the regular MDI as is employed in a word processor application, this application's MDI works in the same manner as is employed in a database application. A different child window is provided for editing each deployment table, each two-dimensional table (chart), and quality planning. One single application can handle only one document of QFD project at a time, but one can work on another project by launching the application in another process. This is realized as shown in Figure 2.

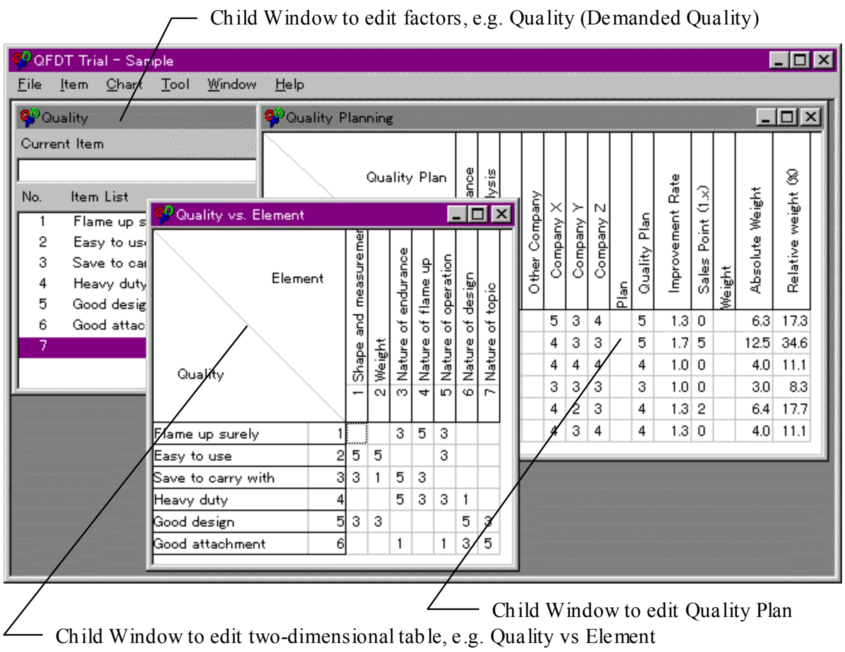

Figure 2 A Screen Image of the Software Tool

System Requirement

At this time, we only provide the software tool, which work on Microsoft Windows 95 and NT 4.0 or higher versions (*). The evaluation versions of the software tool are implemented both in Japanese and English environments. The development project of QFD software tool is being progressed only for the purpose of research and study. If you would like to use and evaluate the tool, please apply to the following web site: http://colie.esi.yamanashi.ac.jp/qfd/.

Usage of the Software Tool

Basic Operations

The basic operations of this software tool include the following operations:

(1) Composing the deployment table. Each deployment table, such as demanded quality (Quality) or quality element (Element), can be composed by clicking "Items" from the main menu and selecting the appropriate items from the pull-down menu. Through this table you can enter the description of each item and set its hierarchical level.

(2) Composing the two-dimensional table. Each two-dimensional table can be composed after each deployment tables combining it. For example, to compose the quality table, we have to first compose the demanded quality and the quality element deployment table. Each two-dimensional table can be composed by clicking "Chart" from the main menu and then selecting the items for its row and column from the pull-down menu and submenu. Through this table you can enter the degrees of correspondence between items by typing the number from 0 to 9. The cell whose degree of correspondence is zero is identical to the blank one.

(3) Composing the quality plan table. The quality plan can be composed after composing the demanded quality deployment table. Click "Chart" and select "Quality Planning" to open the child window for composing the quality plan table. Items usually used in the quality plan such as degree of importance, competitive comparison, plan and weights are fixed and can not be changed, but items in competitive comparison can be skipped if not necessary. Improvement rate, absolute weight and its relative weight are automatically calculated. Through this table, you can enter the number representing the evaluation value. You are requested to enter one decimal digit for sales point, e.g. you need to enter just "5" instead of entering "1.5" of sales point.

(4) Converting the weight of each item. After composing the quality plan table and calculating the weight of each demanded quality item, you can convert this weight, e.g. to the quality element item. For example, to convert the weight of demanded quality item to quality element item, first open the two-dimensional table regarding the quality table. Then, by clicking "Tool" from the main menu and selecting "Convert Weight" from the pull-down menu, you can convert those weights. Each weight is converted using the independent scoring method and automatically converted to a percentage of the total value.

(*) Windows® is a registered trademark of Microsoft Corporation.

Structuralization Using QM3

Here, we will show you how to perform structuralization using QM3 using this software tool. As described above, this method can be applied to each two-dimensional table excluding the quality plan table. The following procedures are the basic operation for this method:

(1) Open the child window of the two-dimensional table.

(2) Select the arbitrarily specified region, which QM3 should be applied to, by highlighting the cells using the regular mouse dragging operation or the cursor key.

(3) Click "Tool" from the main menu and select "Quantification Method Type 3" from the pull-down menu to start the analyzing. If the analyzing is successfully finished, a dialog box with the title "QM3 Result (A vs. B)" (A is name of the deployment table in row and B is name of the deployment table in column of the two-dimensional table), will be appeared as shown in Figure 3. This dialog box shows the eigen values and scores as the result of the QM3 analysis.

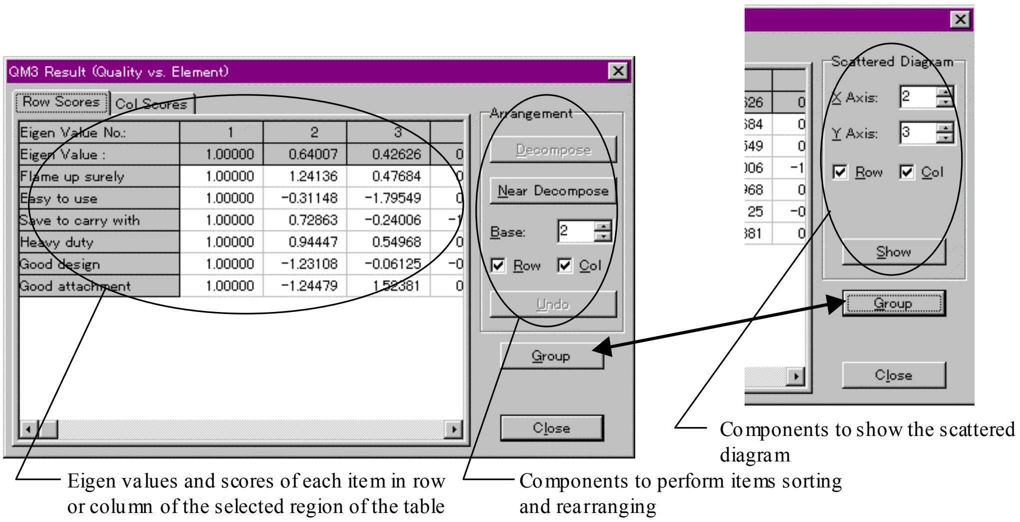

Figure 3 A Screen Image of QM3 Result Dialog Box

Dialog box showing the result of QM3 analysis, consists of the following two main parts:

(1) Part consists of the eigen values and scores correspond to each item in row and column of the selected region of the two dimensional-table. You can click the tab showing "Row Scores" and "Col Scores" to toggle between row and column.

(2) Part consists of the components, such as buttons and edit-boxes showed in group-box, which can be used to perform two operations regarding the result of the analysis. One, which in the group-box called "Arrangement", is for sorting or rearranging the items, and the other, which in the group-box called "Scattered Diagram", is for showing the scattered diagram. You can click the "Group" button to toggle between these two group-boxes to perform the respective operation.

As described in the appendix, the all scores belong to the first eigen value (which always equal to 1.00000) are always equal to 1 and they are meaningless. However, depending on the structure of the table, the eigen values equal to 1 may exist more than one. For n = number of eigen values that equal to 1, this indicates there are n independent parts exist in the selected region, which can be totally decomposed. Decomposition of the table can be performed by clicking the "Decompose" button in the "Arrangement" group-box. The near decomposition can only be performed if number of eigen value that equals to 1 exists only one. The near decomposition can be performed with the following steps within the "Arrangement" group-box:

(1) Select which scores to be used to sort or rearrange the items.

(2) Select which items of row or column or both to sort or rearrange.

(3) Click the "Near Decompose" button to reflect the result to the original table.

Moreover, to show the scattered diagram regarding the scores, first toggle the operation group-box to "Scattered Diagram". Then, the following steps should be performed before showing the diagram:

(1) Select which scores to be put in X-axis and Y-axis.

(2) Select which items of row or column or both to be plotted.

(3) Click the "Show" button to show the scattered diagram.

The scattered diagram will be shown as like in Figure 4. The points correspond to each item will be plotted in the plotting area. This plotting area can be zoomed in or out to see the arbitrarily specific area more detail. Such operation can be performed by showing the pop-up menu and selecting the appropriate command.

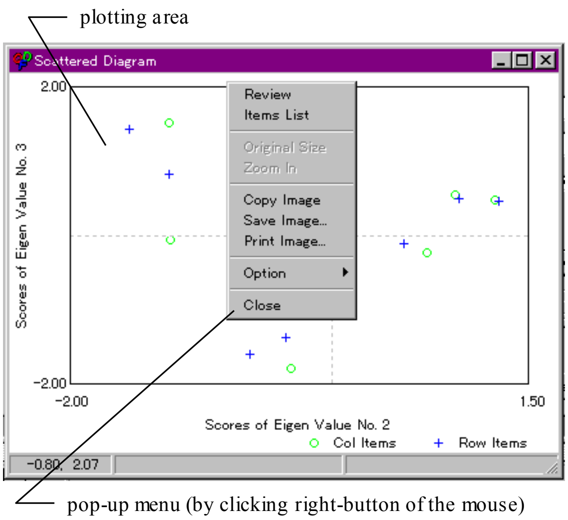

Figure 4 A Screen Image of Scattered Diagram

Concluding Remarks

Various kinds of tables have been used in QFD processes. Before developing the software tool, we standardize them into three types as follow: (1) A one-dimensional table, (2) A two-dimensional table combining two one-dimensional tables, and (3) A two-dimensional used only for quality planning. This simplification of tables is reflected in designing the software tool. Furthermore, this software also can support the user to perform structuralization using QM3, which its significance has been already reported by Shindo (1993), and Xiong and Shindo (1995), to obtain a well-structured and more objective two-dimensional table.

There are over three hundred people from various countries as wells as from Japan have already evaluated this software. We are now working for the improvement of this software tool and reflecting the feedback given by our evaluators.

Currently, this software tool works on a standalone personal computer environment. We are now planning to make it works in network environment, so various people organized in one or more workgroup can share the software and/or the data.

Acknowledgements

The authors wish to thank over three hundred evaluators from various countries as well as from Japan for their valuable comments to improve the tool.

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Appendices

Concept and Example of Quantification Method of Type 3

Quantification Method of Type 3 (QM3) is one of methods for analyzing an inside-structure, which may exist in a two-dimensional table. Here, we consider the following data given in the form of Table 1, where X_i's and Y_j's are items, and x_i's and y_j's are the respective scores, p_{ij} is the relative frequency or the degree of correspondence between X_i and Y_j, $p_{i\cdot}$ and $p_{\cdot j}$ indicate the i-th row sum and the j-th column sum respectively.

Table 1 A Two-Dimensional Table for QM3

items		Y_1	Y_2	$\cdots$	Y_j	$\cdots$	Y_n	
	scores	y_1	y_2	$\cdots$	y_j	$\cdots$	y_n	
X_1	x_1	p_{11}	p_{12}	$\cdots$	p_{1j}	$\cdots$	p_{1n}	$p_{1\cdot}$
X_2	x_2	p_{21}	p_{22}	$\cdots$	p_{2j}	$\cdots$	p_{2n}	$p_{2\cdot}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$		$\vdots$	$\vdots$
X_i	x_i	p_{i1}	p_{i2}	$\cdots$	p_{ij}	$\cdots$	p_{in}	$p_{i\cdot}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\ddots$	$\vdots$	$\vdots$
X_m	x_m	p_{m1}	p_{m2}	$\cdots$	p_{mj}	$\cdots$	p_{mn}	$p_{m\cdot}$
		$p_{\cdot 1}$	$p_{\cdot 2}$	$\cdots$	$p_{\cdot j}$	$\cdots$	$p_{\cdot n}$	1

Table 1 contains two kinds of items, which are not relevant to quantity. However, QM3 analyzes the structure existing between X_i's and Y_j's by allocating scores to each items of x_i's and y_j's.

Without loosing generality, we can put expectations of x_i's and y_j's to the zero respectively, i.e.,

$$\sum_i x_i\, p_{i\cdot} = 0 \qquad (1)$$

$$\sum_j y_j\, p_{\cdot j} = 0 \qquad (2)$$

Furthermore, variances are put to be unity:

$$\sum_i x_i^{\ 2}\ p_{i\cdot} = 1 \tag{3}$$

$$\sum_j y_j^{\ 2}\ p_{\cdot j} = 1 \tag{4}$$

Then, the correlation coefficient ρ is defined by:

$$\rho = \sum_i \sum_j x_i\ y_j\ p_{ij} \tag{5}$$

QM3 determines the values of scores so that the correlation coefficient ρ becomes maximum subject to the conditions both the equations (3) and (4). Then, introducing Lagrange multipliers λ_1 and λ_2, we maximize the following quantity Q.

$$Q = \sum_i \sum_j x_i y_j p_{ij} - \frac{\lambda_1}{2}\left(\sum_i x_i^{\ 2} p_{i\cdot} - 1\right) - \frac{\lambda_2}{2}\left(\sum_j y_j^{\ 2} p_{\cdot j} - 1\right) \tag{6}$$

The partial derivatives with respect to x_i and y_j yield the following equations respectively.

$$\frac{\partial Q}{\partial x_i} = \sum_j y_j p_{ij} - \lambda_1 x_i p_{i\cdot} = 0 \tag{7}$$

$$\frac{\partial Q}{\partial y_j} = \sum_i x_i p_{ij} - \lambda_2 y_j p_{\cdot j} = 0 \tag{8}$$

Multiplying equation (7) by x_i and taking summation with respect to suffix i, we have

$$\sum_i \sum_j x_i y_j p_{ij} - \lambda_1 \sum_i x_i^{\ 2} p_{i\cdot} = 0 \tag{9}$$

Similarly, multiplying equation (8) by y_j and taking summation with respect to suffix j, we have

$$\sum_i \sum_j x_i y_j p_{ij} - \lambda_2 \sum_j y_j^{\ 2} p_{\cdot j} = 0 \tag{10}$$

From equation (3) and (4), and (9) and (10), we obtain that $\lambda_1 = \lambda_2$, and we put them λ. Thus, from equation (5), we know that $\lambda = \rho$.

Equation (7) is written by

$$x_i = \frac{\sum_j y_j p_{ij}}{\lambda\, p_{i\cdot}} \tag{11}$$

Substituting (11) into (8), we have

$$\sum_i \frac{\sum_j y_k p_{ij}}{p_{i\cdot}} p_{ij} - \lambda^2 y_j p_{\cdot j} = 0 \tag{12}$$

Here, we introduce the following notations:

$$\boldsymbol{x} = \begin{pmatrix} x_1 & x_2 & \cdots & x_m \end{pmatrix}^T \tag{13}$$

$$\boldsymbol{y} = \begin{pmatrix} y_1 & y_2 & \cdots & y_n \end{pmatrix}^T \tag{14}$$

$$\boldsymbol{P} = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{m1} & p_{m2} & \cdots & p_{mn} \end{bmatrix} \tag{15}$$

$$\boldsymbol{P}_x = \begin{bmatrix} p_{1\cdot} & 0 & \cdots & 0 \\ 0 & p_{2\cdot} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p_{m\cdot} \end{bmatrix}, \ \boldsymbol{P}_x{}^k = \begin{bmatrix} p_{1\cdot}{}^k & 0 & \cdots & 0 \\ 0 & p_{2\cdot}{}^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p_{m\cdot}{}^k \end{bmatrix} \tag{16}$$

$$\boldsymbol{P}_y = \begin{bmatrix} p_{\cdot 1} & 0 & \cdots & 0 \\ 0 & p_{\cdot 2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p_{\cdot n} \end{bmatrix}, \ \boldsymbol{P}_y{}^k = \begin{bmatrix} p_{\cdot 1}{}^k & 0 & \cdots & 0 \\ 0 & p_{\cdot 2}{}^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p_{\cdot n}{}^k \end{bmatrix} \tag{17}$$

Then, equation (12) is given as follows:

$$\boldsymbol{P}^T \boldsymbol{P}_x^{-1} \boldsymbol{P}\, \boldsymbol{y} - \lambda^2\, \boldsymbol{P}_y\, \boldsymbol{y} = 0 \tag{18}$$

After rearranging equation (18) as follows, this indicates an eigen value problem whose eigen value is λ^2.

$$\left(\boldsymbol{P}_y^{-1/2} \boldsymbol{P}^T \boldsymbol{P}_x^{-1} \boldsymbol{P}\, \boldsymbol{P}_y^{-1/2} - \lambda^2\, \boldsymbol{I} \right) \boldsymbol{P}_y^{1/2}\, \boldsymbol{y} = 0 \tag{19}$$

Therefore, putting

$$\boldsymbol{v} = \boldsymbol{P}_y^{1/2}\, \boldsymbol{y} \tag{20}$$

and solving the following eigen value problem

$$\left(\boldsymbol{P}_y^{-1/2} \boldsymbol{P}^T \boldsymbol{P}_x^{-1} \boldsymbol{P}\, \boldsymbol{P}_y^{-1/2} - \lambda^2\, \boldsymbol{I} \right) \boldsymbol{v} = 0 \tag{21}$$

we can obtain the score $\boldsymbol{y}$ by

$$\boldsymbol{y} = \boldsymbol{P}_y^{-1/2}\, \boldsymbol{v} \tag{22}$$

Substituting equation (22) into (11) yields score $\boldsymbol{x}$ as follows:

$$\boldsymbol{x} = \frac{1}{\lambda} \boldsymbol{P}_x^{-1/2}\, \boldsymbol{P}\, \boldsymbol{y} \tag{23}$$

Next, let us consider the following example of two-dimensional table, which indicates three persons like each juice.

Table 2 Example of Two-Dimensional Table for QM3

	orange	pineapple	grape
A	∘		
B	∘		∘
C		∘	∘

Table 2 can be rewritten as follows:

Table 3 Example of Two-Dimensional Table for QM3 (Cont'd)

	b_1	b_2	b_3	
a_1	(a_1, b_1)			1
a_2	(a_2, b_1)		(a_2, b_3)	2
a_3		(a_3, b_2)	(a_3, b_3)	2
	2	1	2	5

Put each averages of a and b, i.e.

$$\mathrm{avr}(a) = \left(a_1 + 2a_2 + 2a_3 \right) / 5 = 0 \tag{24}$$

$$\mathrm{avr}(b) = \left(2b_1 + b_2 + 2b_3 \right) / 5 = 0 \tag{25}$$

Variances are also put to be unity:

$$\left(a_1{}^2 + 2a_2{}^2 + 2a_3{}^2 \right) / 5 = 1 \tag{26}$$

$$\left(2b_1{}^2 + b_2{}^2 + 2b_3{}^2 \right) / 5 = 1 \tag{27}$$

Then, the correlation coefficient ρ is given by

$$\rho = \left(a_1 b_1 + a_2 b_1 + a_2 b_3 + a_3 b_2 + a_3 b_3 \right) \tag{28}$$

We maximize the following Q for Lagrange multipliers λ_1 and λ_2:

$$\begin{aligned} Q = & \left(a_1 b_1 + a_2 b_1 + a_2 b_3 + a_3 b_2 + a_3 b_3 \right) / 5 \\ & - \lambda_1 / 2 \left(\left(a_1{}^2 + 2a_2{}^2 + 2a_3{}^2 \right) / 5 - 1 \right) \\ & - \lambda_2 / 2 \left(\left(2b_1{}^2 + b_2{}^2 + 2b_3{}^2 \right) / 5 - 1 \right) \end{aligned} \tag{29}$$

Partial derivatives with respect to scores are as follows:

$$\partial Q / \partial a_1 = \left(b_1 - \lambda_1 a_1 \right) / 5 = 0 \tag{30}$$

$$\partial Q / \partial a_2 = \left(b_1 + b_3 - 2\lambda_1 a_2 \right) / 5 = 0 \tag{31}$$

$$\partial Q / \partial a_3 = \left(b_2 + b_3 - 2\lambda_1 a_3 \right) / 5 = 0 \tag{32}$$

$$\partial Q / \partial b_1 = \left(a_1 + a_2 - 2\lambda_2 b_1 \right) / 5 = 0 \tag{33}$$

$$\partial Q / \partial b_2 = \left(a_3 - \lambda_2 b_2 \right) / 5 = 0 \tag{34}$$

$$\partial Q / \partial b_3 = \left(a_2 + a_3 - 2\lambda_2 b_3 \right) / 5 = 0 \tag{35}$$

From the fact that

$$a_1 \times (30) + a_2 \times (31) + a_3 \times (32) = \left(\left(a_1 b_1 + a_2 b_1 + a_2 b_3 + a_3 b_2 + a_3 b_3 \right) - \lambda_1 \left(a_1{}^2 + 2a_2{}^2 + 2a_3{}^2 \right) \right) / 5 \tag{36}$$

and that

$$b_1 \times (33) + b_2 \times (34) + b_3 \times (35) = \left(\left(a_1 b_1 + a_2 b_1 + a_2 b_3 + a_3 b_2 + a_3 b_3 \right) - \lambda_2 \left(2b_1{}^2 + b_2{}^2 + 2b_3{}^2 \right) \right) / 5 \tag{37}$$

and from equation (26) and (27), we put $\lambda_1 = \lambda_2 = \lambda$. Thus, from equation (28), we know that $\lambda = \rho$.

Equations (30), (31) and (32) yield

$$a_1 = b_1 / \lambda \tag{38}$$

$$a_2 = \left(b_1 + b_3 \right) / (2\lambda) \tag{39}$$

$$a_3 = \left(b_2 + b_3 \right) / (2\lambda) \tag{40}$$

Substituting above three equations into (33), (34) and (35), we have

$$b_1 / \lambda + \left(b_1 + b_3 \right) / (2\lambda) - 2\lambda b_1 = 0 \tag{41}$$

$$\left(b_1 + b_3 \right) / (2\lambda) - \lambda b_2 = 0 \tag{42}$$

$$\left(b_1 + b_3 \right) / (2\lambda) + \left(b_2 + b_3 \right) / (2\lambda) - 2\lambda b_3 = 0 \tag{43}$$

These can be arranged as follows:

$$\begin{array}{lllll} (3/4) b_1 - \lambda^2 b_1 & & + (1/4) b_3 & = & 0 \\ & (1/2) b_2 - \lambda^2 b_2 & + (1/2) b_3 & = & 0 \\ (1/4) b_1 & + (1/4) b_2 & + (1/2) b_3 - \lambda^2 b_3 & = & 0 \end{array} \tag{44}$$

Finally we can get the following equations:

$$\begin{array}{lllll}
(3/4)\left(\sqrt{2}\,b_1\right)-\lambda^2\left(\sqrt{2}\,b_1\right) & & +(1/4)\left(\sqrt{2}\,b_3\right) & = & 0 \\
 & \left((1/2)-\lambda^2\right)(b_2) & +\left(1/\left(2\sqrt{2}\right)\right)\left(\sqrt{2}\,b_3\right) & = & 0 \\
(1/4)\left(\sqrt{2}\,b_1\right) & +\left(1/\left(2\sqrt{2}\right)\right)(b_2) & +\left((1/2)-\lambda^2\right)\left(\sqrt{2}\,b_3\right) & = & 0
\end{array} \tag{45}$$

These can be expressed by the following matrix expression:

$$\begin{bmatrix} (3/4)-\lambda^2 & & +(1/4) \\ & (1/2)-\lambda^2 & +1/\left(2\sqrt{2}\right) \\ (1/4) & +1/\left(2\sqrt{2}\right) & +(1/2)-\lambda^2 \end{bmatrix} \begin{bmatrix} \sqrt{2}\,b_1 \\ b_2 \\ \sqrt{2}\,b_3 \end{bmatrix} = \mathbf{0} \tag{46}$$

This can be solved with respect to λ^2 as follows:

$$16\left(\lambda^2\right)^3-28\left(\lambda^2\right)^2+13\left(\lambda^2\right)-1=\left(\left(\lambda^2\right)-1\right)\left(16\left(\lambda^2\right)-12\left(\lambda^2\right)+1\right)=0 \tag{47}$$

After all, we have three roots:

$$\begin{aligned} \lambda^2 &= 1 \\ \lambda^2 &= \left(3\pm\sqrt{5}\right)/8 \\ &\approx 0.65451, \text{ and} \\ &\approx 0.09549 \end{aligned} \tag{48}$$

When $\lambda^2 = 1$, $b_1 = b_2 = b_3 = 1/\sqrt{5}$. This contradicts the assumption that the average of scores is equal to zero. Thus, we exclude this case because of meaningless.

Substituting the second and the third cases of λ^2 into equation (46), we have:

$$\left(\left(-3\pm\sqrt{5}\right)/8\right)\left(\sqrt{2}\,b_1\right)-\left(\sqrt{2}\,b_3\right)/4=0 \tag{49}$$

$$\left(\left(-1\pm\sqrt{5}\right)/8\right)b_2-\left(\sqrt{2}\,b_3\right)/\left(2\sqrt{2}\right)=0 \tag{50}$$

$$\left(\sqrt{2}\,b_1\right)/(-4)-b_2/\left(2\sqrt{2}\right)-\left(\left(-1\pm\sqrt{5}\right)/8\right)\left(\sqrt{2}\,b_3\right)=0 \tag{51}$$

Therefore, b_2 and b_3 are expressed by using b_1 respectively by

$$\begin{aligned} b_2 &= -\left(-1\pm\sqrt{5}\right)b_1 \\ b_3 &= \left(\left(-3\pm\sqrt{5}\right)/2\right)b_1 \end{aligned} \tag{52}$$

Substitution of above equations into equation (27) yields

$$b_1^{\,2}=\left(3\pm\sqrt{5}\right)/4 \tag{53}$$

Then, we can obtain the following scores with determining $b_1 > 0$.

$$\begin{array}{lcrr} b_1 & \approx & 1.14412 & 0.43702 \\ b_2 & \approx & -1.41421 & 1.41421 \\ b_3 & \approx & -0.43702 & -1.14412 \end{array} \tag{54}$$

Finally, we can obtain a_1, a_2 and a_3 by substituting above b_1, b_2 and b_3 into equations (38), (39) and (40) respectively. Then, we obtained the following scores:

$$\begin{array}{lcrr} a_1 & \approx & 1.41421 & 1.41421 \\ a_2 & \approx & 0.43702 & -1.14412 \\ a_3 & \approx & -1.14412 & 0.43702 \end{array} \tag{55}$$