Applied Mathematical Sciences, Vol. 12, 2018, no. 2, 57 - 68
HIKARI Ltd, www.m-hikari.com
https://doi.org/10.12988/ams.2018.712361

A Comparative Study of Robust t Linear Mixed Models with Application to Household Consumption Per Capita Expenditure Data

Azka Ubaidillah

Department of Statistics, Bogor Agricultural University, Bogor, Indonesia
Institute of Statistics, Jakarta, Indonesia

Khairil Anwar Notodiputo*

Department of Statistics, Bogor Agricultural University, Bogor, Indonesia
*Corresponding author

Anang Kurnia

Department of Statistics, Bogor Agricultural University, Bogor, Indonesia

I Wayan Mangku

Department of Mathematics, Bogor Agricultural University, Bogor, Indonesia

Abstract

The linear mixed models (LMMs) are widely used for data analysis to account fixed effects and random effects in Gaussian response models. In LMMs, the random effects and the within-subject errors have been assumed to be normally distributed but in practice, such an assumption could easily be violated due to the presence of atypical data. Motivated by a concern of sensitivity to potential outliers or data with longer-than-normal tails, many researchers have developed robust LMMs using t distribution (abbreviated as tLMM). This paper discussed the comparison between the LMMs and the tLMMs especially from perspectives of the fitness and robustness of the models. The application of tLMM to household consumption per capita expenditure data was also demonstrated in this paper. The results of this study showed that the tLMMs provided better estimates than LMMs in term of performance and robustness. Furthermore, it was also showed that the best model to handle outliers was found to be the tLMM with the smallest degrees of freedom.

Keywords: household consumption per capita expenditure, linear mixed models, robustness, student-t distribution

1 Introduction

The linear mixed model (LMM), originally proposed by [7], has been widely used for the data analysis. The popularity of such a model arises from its implementation through commonly available software, e.g., SAS [13], procedure MIXED, and R [12], library NLME. Comprehensive reviews that cover methodological and computational aspects of the LMM are contained in many books, among of them are written by [2] and [4].

In the framework of LMMs, the random effects and the within-subject errors are assumed to the normal distribution. However, such an assumption is not always satisfied because of the presence of atypical data. To remedy this weakness, many robust methods have been developed to be less sensitive to outliers. [19, 8, 6, 15, 16] investigated the use of the t distribution in place of the normal for robust regression. The value of v (called the degrees of freedom), which controls the thickness of the tails of the distribution, is directly related to the degree of robustness, the smaller of v yields the higher of robustness [16]. A robustness against outliers of t linear mixed model (tLMM) through an application to orthodontic data and extensive simulations was proposed by [11]. Further work in this direction is studied by [9, 10], and [14].

As well known, household income is an economic indicator that is often used for measuring the prosperity and well-being. However, household income is generally very difficult to be measured accurately, especially in developing countries. Basically, household income and household expenditure are not the same things. But such relationships between those two are very strong. [1] stated that consumption expenditure is more reliable than income as an indicator of a household permanent income because it does not vary as much as income in the short term. For those reasons, household expenditure patterns approach is then widely used to analyze the pattern of household income [5].

This paper presents a comparative study of robustness in tLMM on household expenditure data by employing many different degrees of freedom (v). In Section 2, we describe a dataset for models application i.e household consumption per capita expenditure in Jambi City in 2011. In Section 3, we define notation and model formulation. The score vector and Fisher information matrix are derived. In Section 5, we show the results of models application and its simulation to demonstrate the robustness of tLMM with the household consumption per capita expenditure data preliminarily analyzed in Section 2. Finally, Section 6, we state some concluding remarks.

2 Data

Household consumption per capita expenditure (HCPE) data have been collected by the National Socioeconomic Surveys (Susenas) which have been conducted regularly by Statistics Indonesia (BPS). In this paper, we use the HCPE data of Jambi City, Indonesia, in 2011 which have been collected quarterly in a year. The sample size of data is 575 household which distributed almost the same in each quarterly months. The dependent variable used in this study is HCPE (Y). The several household attributes with fixed effects (X) and random effect (Z) that are considered as having affected the household consumption per capita expenditure are shown in Table 1.

Household expenditure distribution has a shape that closes to a right-skewed distribution such as lognormal or loglogistic. In this study, we employ the Boxcox transformation in order to obtain symmetrical histogram as shown in Figure 1. The lambda of Boxcox transformation is $\lambda = -0.5$, so the data transformation is $Y^* = Y^{(-0,5)}$.

Table 1: Variables used in Model Applications

Variables	Description
Y	Household consumption per capita expenditure
X_1	Household size
X_2	= 1, if the level of household head education is not passed is not passed elementary school = 0, if others
X_3	= 1, if the level of household head education is not passed junior high school = 0, if others
X_4	Percentage of household member who is working
X_5	= 1, if the type of house floor is from tile = 0, if others
X_6	= 1, if the type of cooking fuel is from electricity or LPG = 0, if others
Z_1	Quarterly months (1,2,3,4)

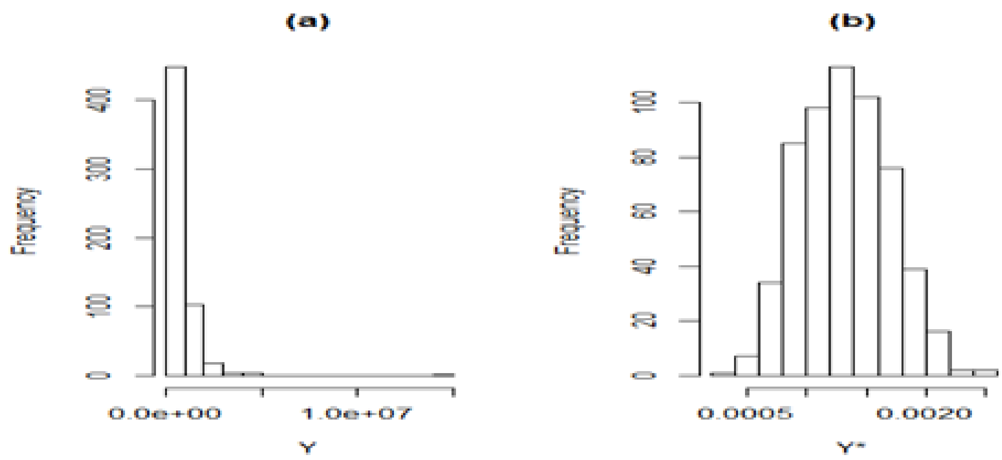

Figure 1: Histogram of (a) original data Y, and (b) transformed data Y^*

3 Methods

Let $y_i = (y_{i1}, \dots, y_{in_i})^T$ be dependent variable with $i = 1, \dots, N$ and $t = 1, \dots, n_i$. Consider e_i the within error corresponding to y_i. Let X_i be covariates of $n_i \times q_1$ design matrix for fixed effects, and let Z_i be an $n_i \times q_2$ design matrix for random effects. Then the form of tLMM be [17]:

$$\mathbf{y}_i = \mathbf{X}_i\boldsymbol{\beta} + \mathbf{Z}_i\mathbf{b}_i + \mathbf{e}_i, \tag{1}$$

with

$$\begin{pmatrix}\mathbf{b}_i\\ \mathbf{e}_i\end{pmatrix} \sim t_{(q_2+n_i)}\left[\begin{pmatrix}\mathbf{0}\\ \mathbf{0}\end{pmatrix}, \begin{pmatrix}\mathbf{D} & \mathbf{0}\\ \mathbf{0} & \mathbf{R}\end{pmatrix}\right],$$

where $\boldsymbol{\beta} = \left(\beta_0^T, \beta_1^T, \dots, \beta_{q_1-1}^T\right)^T$ is the regression parameter for fixed effects, and $\mathbf{b}_i = \left(b_1^T, \dots, b_{q_2}^T\right)^T$ is q_2-vector of random effects. We assume that $\mathbf{D}$ is $q_2 \times q_2$ symmetric positive-definite matrix of unstructured covariance (σ_b^2) in random effects, and $\mathbf{R}_i$ is $n_i \times n_i$ structured covariance matrix in error components. We assume that the joint distributions of $\mathbf{b}_i$ and $\mathbf{e}_i$ are independent, and we take $v_i = v$ for all i. For the within-subject error $\mathbf{e}_i$, we assume has identically, independently and has distribution $t_{(n_i)}(0, \sigma_e^2\mathbf{I}_i, v)$.

Under this consideration, the response variable of (1) is assumed has distribution

$$\mathbf{y}_i \sim t_{n_i}(\mathbf{X}_i\boldsymbol{\beta}, \boldsymbol{\Lambda}_i, v), \tag{2}$$

where

$$\boldsymbol{\Lambda}_i = \boldsymbol{\Lambda}_i(\mathbf{D}) = \mathbf{Z}_i\mathbf{D}\mathbf{Z}_i^T + \sigma_e^2\mathbf{I}_i$$

are implicit functions depending on element of $\mathbf{D}$ and σ_e^2. If $v > 1$, then $\mathbf{X}_i\boldsymbol{\beta}$ is the mean of $\mathbf{y}_i$, and if $v > 2$, then $v(v-2)^{-1}\boldsymbol{\Lambda}_i$ is variance covariance matrix of $\mathbf{y}_i$.

Let $\boldsymbol{\Delta}_i$ be the Mahalanobis distance between $\mathbf{y}_i$ and $\mathbf{X}_i\boldsymbol{\beta}$, then we have

$$\boldsymbol{\Delta}_i = \boldsymbol{\Delta}_i(\boldsymbol{\beta}, \mathbf{D}) = \boldsymbol{\epsilon}_i^T \boldsymbol{\Lambda}_i^{-1} \boldsymbol{\epsilon}_i$$

where $\boldsymbol{\epsilon}_i = \mathbf{y}_i - \mathbf{X}_i\boldsymbol{\beta}$. Let $\boldsymbol{\theta} = \left(\boldsymbol{\beta}^T, \boldsymbol{\alpha}^T\right)^T$ be the vector of unknown parameters, where $\boldsymbol{\alpha} = (\boldsymbol{\omega}^T, v)^T$ with $\boldsymbol{\omega} = \left(\text{vech}(\mathbf{D})^T, \text{vech}(\sigma_e^2\mathbf{I}_i\right)^T$. Given independent observations $Y_1, \ldots, Y_N$, we can write the log-likelihood function of (2) as $\ell = \sum_{i=1}^N l_i$, where

$$\begin{aligned} l_i = &\log\left(\Gamma\left(\frac{v+n_i}{2}\right)\right) - \log\left(\Gamma\left(\frac{v}{2}\right)\right) - \frac{n_i}{2}\log(\pi v) - \frac{1}{2}\log|\boldsymbol{\Lambda}_i| \\ &- \frac{v+n_i}{2}\log\left(1 + \frac{\boldsymbol{\Delta}_i}{v}\right). \end{aligned} \tag{3}$$

We can obtained the score vector $\mathbf{s}_{\boldsymbol{\theta}} = \left(\mathbf{s}_{\boldsymbol{\beta}}^T, \mathbf{s}_{\boldsymbol{\alpha}}^T\right)^T$ and the Fisher information matrix $\mathbf{I}_{\boldsymbol{\theta}\boldsymbol{\theta}}$ by computing the first and the second derivatives of (3). The score vectors $\mathbf{s}_{\boldsymbol{\theta}}$ are

$$\begin{aligned} \mathbf{s}_{\boldsymbol{\beta}} &= \sum_{i=1}^N (v+n_i)\frac{\mathbf{X}_i^T\boldsymbol{\Lambda}_i^{-1}\boldsymbol{\epsilon}_i}{v+\boldsymbol{\Delta}_i}, \\ \mathbf{s}_{\sigma_e^2} &= -\frac{\sum_{i=1}^N}{2\sigma_e^2} + \frac{1}{2\sigma_e^2}\sum_{i=1}^N (v+n_i)\frac{\boldsymbol{\Delta}_i}{v+\boldsymbol{\Delta}_i}, \\ \mathbf{s}_v &= \frac{1}{2}\sum_{i=1}^N\left[\phi\left(\frac{v+n_i}{2}\right) - \phi\left(\frac{v}{2}\right) - \frac{n_i}{2} - \log\left(1+\frac{\boldsymbol{\Delta}_i}{v}\right) + \frac{v+n_i}{v}\frac{\boldsymbol{\Delta}_i}{v+\boldsymbol{\Delta}_i}\right], \\ [\mathbf{s}_{\boldsymbol{\omega}}]_r &= -\frac{1}{2}\sum_{i=1}^N\left[\text{tr}\left(\boldsymbol{\Lambda}_i^{-1}\dot{\boldsymbol{\Lambda}}_{ir}\right) - (v+n_i)\left(\frac{\boldsymbol{\epsilon}_i^T\boldsymbol{\Lambda}_i^{-1}\dot{\boldsymbol{\Lambda}}_{ir}\boldsymbol{\Lambda}_i^{-1}\boldsymbol{\epsilon}_i}{v+\boldsymbol{\Delta}_i}\right)\right], \end{aligned} \tag{4}$$

where

$$\dot{\boldsymbol{\Lambda}}_{ir} = \left(\frac{\partial\boldsymbol{\Lambda}_i}{\partial\boldsymbol{\omega}_r}\right), \quad \text{for} \quad r = 1, \ldots, g; g = \left(\frac{q_2^2 + q_2 + 2}{2}\right)$$

and $\phi(x) = \frac{d}{dx}\log(\Gamma(x))$ denotes the digamma function.

The Fisher information, obtained by negative expectation of the second derivative of (3), has the following forms:

$$
\begin{aligned}
\Im_{\boldsymbol{\beta}\boldsymbol{\beta}} &= \sum_{i=1}^{N}\left[\frac{v+n_i}{\sigma^2\left(v+n_i+2\right)}\right]\mathbf{X}_i^T\boldsymbol{\Lambda}_i^{-1}\mathbf{X}_i,\\
\Im_{\boldsymbol{\beta}\sigma} &= \mathbf{0}_{q_1\times 1},\quad \Im_{\boldsymbol{\beta}v} = \mathbf{0}_{q_1\times 1},\quad \Im_{\boldsymbol{\beta}\boldsymbol{\omega}} = \mathbf{0}_{q_1\times g},\\
\Im_{\sigma_e^2\sigma_e^2} &= \frac{v}{2\sigma_e^4}\sum_{i=1}^{N}\frac{n_i}{v+n_i+2},\\
[\Im_{\sigma_e^2\boldsymbol{\omega}}]_r &= \frac{v}{2\sigma_e^2}\sum_{i=1}^{N}\frac{1}{v+n_i+2}\mathrm{tr}\left(\boldsymbol{\Lambda}_i^{-1}\dot{\boldsymbol{\Lambda}}_{ir}\right),\\
\Im_{vv} &= \frac{1}{4}\sum_{i=1}^{N}\left[\psi\left(\frac{v}{2}\right)-\psi\left(\frac{v+n_i}{2}\right)-\frac{2(v+2)}{v(v+n_i+2)}-\frac{2}{v}+\frac{4}{v+n_i}\right],\\
[\Im_{v\boldsymbol{\omega}}]_r &= -\sum_{i=1}^{N}\left[\frac{1}{(v+n_i)(v+n_i+2)}\mathrm{tr}\left(\boldsymbol{\Lambda}_i^{-1}\dot{\boldsymbol{\Lambda}}_{ir}\right)\right],\\
[\Im_{\boldsymbol{\omega}\boldsymbol{\omega}}]_{rs} &= \frac{1}{2}\sum_{i=1}^{N}\left[\frac{1}{v+n_i+2}(v+n_i)\mathrm{tr}\left(\boldsymbol{\Lambda}_i^{-1}\dot{\boldsymbol{\Lambda}}_{ir}\boldsymbol{\Lambda}_i^{-1}\dot{\boldsymbol{\Lambda}}_{is}\right)-\mathrm{tr}\left(\boldsymbol{\Lambda}_i^{-1}\dot{\boldsymbol{\Lambda}}_{ir}\right)\mathrm{tr}\left(\boldsymbol{\Lambda}_i^{-1}\dot{\boldsymbol{\Lambda}}_{ir}\right)\right],
\end{aligned}
\tag{5}
$$

for $r = 1, \dots, g$, where $\psi(x) = \frac{d^2}{dx^2}\log\left(\Gamma(x)\right)$ denotes the trigamma function.

To obtain the maximum likelihood (ML) estimates, we employ the Fisher scoring algorithm. Under some regularity conditions, the asymptotic covariance matrix estimates can be computed by substituting the ML estimates, $\hat{\boldsymbol{\theta}} = (\hat{\boldsymbol{\beta}}, \hat{\boldsymbol{\alpha}})$, into the inverse of the Fisher information matrix of (5). The asymptotic covariance matrix of $\hat{\boldsymbol{\beta}}$ and $\hat{\boldsymbol{\alpha}}$ can be formed as

$$
\begin{aligned}
\mathrm{var}(\hat{\boldsymbol{\beta}}) &= \hat{\Im}_{\boldsymbol{\beta}\boldsymbol{\beta}}^{-1} = \hat{\sigma}^2\left(\sum_{i=1}^{N}\frac{v+n_i}{v+n_i+2}\mathbf{X}_i^T\boldsymbol{\Lambda}_i^{-1}\mathbf{X}_i\right)^{-1},\\
\mathrm{var}(\hat{\boldsymbol{\alpha}}) &= \hat{\Im}_{\boldsymbol{\alpha}\boldsymbol{\alpha}}^{-1}.
\end{aligned}
$$

The ML estimates of variance components are biased downward in finite samples size. REML produces unbiased estimating equations for the variance components and corrects for the loss of degrees-of-freedom incurred in estimating the fixed effects. As noted by [3], REML can be viewed as the Bayesian principle of marginal inference by adopting the prior distribution $\pi(\boldsymbol{\beta}, \boldsymbol{\alpha}) \propto 1$ and Laplaces method as in [18]. Under this consideration, the t-REML likelihood function can be approximated by $L_R(\boldsymbol{\alpha}) = \int L(\boldsymbol{\beta}, \boldsymbol{\alpha})d\boldsymbol{\beta} \approx L_R^*(\boldsymbol{\alpha})$, where

$$
L_R^*(\boldsymbol{\alpha}) = (\sigma^2\pi v)^{-n/2}\left|\sum_{i=1}^{N}\mathbf{X}_i^T\mathbf{H}_i\mathbf{X}_i\right|^{-1/2}\prod_{i=1}^{N}\frac{\Gamma\left(\frac{v+n_i}{2}\right)}{\Gamma\left(\frac{v}{2}\right)}\times|\boldsymbol{\Lambda}_i|^{-1/2}\left(1+\frac{\boldsymbol{\Delta}_i}{v}\right)^{-\frac{v+n_i}{2}},
$$

where

$$\mathbf{H}_i = (v+n_i)\left[\frac{\mathbf{\Lambda}_i^{-1}}{v+\mathbf{\Delta}_i\left(\hat{\boldsymbol{\beta}}(\boldsymbol{\alpha}),\mathbf{D}\right)} - \frac{2\mathbf{\Lambda}_i^{-1}\hat{\boldsymbol{\epsilon}}_i(\boldsymbol{\alpha})\hat{\boldsymbol{\epsilon}}_i^T(\boldsymbol{\alpha})\mathbf{\Lambda}_i^{-1}}{v+\mathbf{\Delta}_i\left(\hat{\boldsymbol{\beta}}(\boldsymbol{\alpha}),\mathbf{D}\right)^2}\right],$$

with $\hat{\boldsymbol{\epsilon}}_i(\boldsymbol{\alpha}) = \mathbf{y}_i - \mathbf{X}_i\hat{\boldsymbol{\beta}}(\boldsymbol{\alpha})$, and $\hat{\boldsymbol{\beta}}(\boldsymbol{\alpha})$ is obtained by solving the following equation:

$$\sum_{i=1}^{N}(v+n_i)\left[\frac{\mathbf{X}_i^T\mathbf{\Lambda}_i^{-1}(\mathbf{y}_i-\mathbf{X}_i\boldsymbol{\beta})}{v+\mathbf{\Delta}_i\left(\boldsymbol{\beta}(\boldsymbol{\alpha}),\mathbf{D}\right)}\right] = 0. \qquad (6)$$

The approximately REML estimates of $\boldsymbol{\alpha}, \hat{\boldsymbol{\alpha}}_R$, can be obtained by implementing the Newton-Raphson (NR) algorithm with ML estimates as the initial values, and the empirical Bayes estimates of $\boldsymbol{\beta}, \hat{\boldsymbol{\beta}}(\hat{\boldsymbol{\alpha}}_R)$, must be computed at each iteration by solving the equation (6) with $\boldsymbol{\alpha}$ replaced by the current estimate $\hat{\boldsymbol{\alpha}}_R$.

4 Main Results

In this section, we show the results of LMM and tLMM with different degrees of freedom v on the set of Household expenditure data pre-analyzed in Section 2. The tLMM with $v = 1(t_1\text{LMM})$, $v = 3(t_3\text{LMM})$, $v = 5(t_5\text{LMM})$, and the estimate of $v, \hat{v}(t_{\hat{v}}\text{LMM})$ are compared. We model the household consumption per capita expenditure as the linear function of $X_1, \ldots, X_6$ as the fixed effects and Z_1 as the random effect. We set $\boldsymbol{\beta} = (\beta_0, \beta_1, \ldots, \beta_6)^T$ as the parameter regressions for the fixed effects with the corresponding of design matrix $\mathbf{X}_i = [\mathbf{1}_i : \mathbf{X}_{1i} : \ldots : \mathbf{X}_{6i}]$.

Table 2 reports the estimate values of $\boldsymbol{\beta}$ with their standard errors in the parentheses, the estimates of standard deviation of random effect and within subject error. The tLMM performance is better than the LMM of which the standard deviation within subject errors in tLMM are smaller than the LMM. However, the tLMM with estimate degrees of freedom ($\hat{v} = 13.066$) has the standard errors of parameter regressions for fixed effects that slight bigger than the LMM, but the others of tLMM have smaller standard errors of parameter regression than LMM. The tLMM with $v = 1$ has the smallest standard deviation within subject errors and standard errors of parameter regression for fixed effects.

Robustness

We use a simulation study to demonstrate the robustness of the tLMM. The simple way to assess the robustness of the model is to make the Sensitivity Curve $SC_n(y_m)$ as shown in Figure 2. In many situations, $SC_n(y_m)$ will

Table 2: Estimate results of LMM and tLMM

	Parameter	LMM	t_1LMM	t_3LMM	t_5LMM	$t_{\hat{v}}$LMM
Fixed effects	β_0	12.566	11.360	11.940	12.100	12.320
		(0.0537)	(0.0399)	(0.0456)	(0.0467)	(0.0375)
	β_1	0.0737	0.0835	0.0783	0.0771	0.0754
		(0.0080)	(0.0066)	(0.0076)	(0.0077)	(0.0120)
	β_2	0.1277	0.1967	0.1779	0.1700	0.1566
		(0.0395)	(0.0326)	(0.0373)	(0.0381)	(0.0582)
	β_3	0.1378	0.1967	0.1742	0.1672	0.1571
		(0.0273)	(0.0225)	(0.0257)	(0.0263)	(0.0400)
	β_4	-0.0024	-0.0018	-0.0022	-0.0023	-0.0024
		(0.0005)	(0.0004)	(0.0005)	(0.0005)	(0.0077)
	β_5	-0.2047	-0.1890	-0.1975	-0.1999	-0.2053
		(0.0261)	(0.0215)	(0.0246)	(0.0251)	(0.0385)
	β_6	-0.2001	-0.2012	-0.1908	-0.1897	-0.1889
		(0.0265)	(0.0216)	(0.0247)	(0.0252)	(0.0387)
Random effect	σ_b	0.0475	0.0180	0.0190	0.0187	0.0179
Within subject error	σ_e	0.2822	0.1655	0.2186	0.2370	0.2607
Degrees of freedom	v	-	1	3	5	13.066
						(0.444)

converge to the influence function when $n \to \infty$. The procedures for making Sensitivity Curve applied on household consumption per capita expenditure data are:

1. Use the household consumption per capita expenditure data set pre-analyzed in Section 2.

2. Consider the arbitrary value of y_m, hence we set $y_m = \text{seq}(0.0001, 0.003, \text{length} = 1000)$. Replace one observation of the rest response data, Y^*_{575}, by an arbitrary value y_m and count the value of

$$SC_n(y_m) = n\left(T_n\left(x_1, \dots, x_{(n-1)}, y_m\right) - T_{(n-1)}\left(x_1, \dots, , x_{(n-1)}\right)\right),$$

   where $T_n(x)$ denotes the estimator of interest based on the sample X of size n.

3. Plot the value of $SC_n(y_m)$ as the Y-axis and y_m as the X-axis.

Figure 2 exhibits the curves of changes for the estimates of $\boldsymbol{\beta}$ applied on Household expenditure data with plugging influence of outliers. The influence on parameter estimation of the outliers is unbounded in the case of the LMM, whereas it is obviously bounded in the tLMM. More specifically, outliers in the

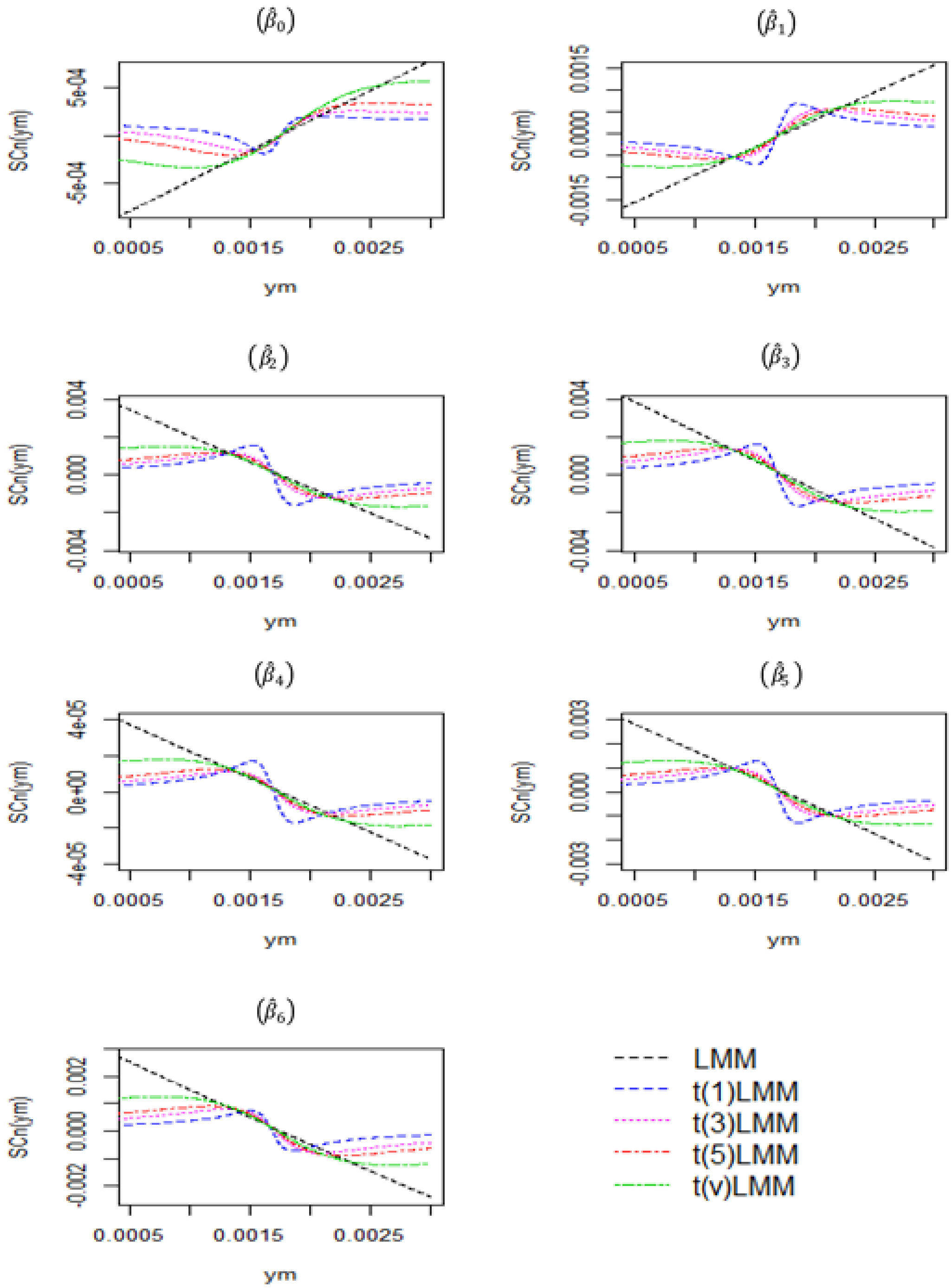

Figure 2: Sensitivity Curve for Parameter Estimates of $\boldsymbol{\beta}$ in LMM and tLMM.

tLMM with smallest degrees of freedom ($v = 1$) shows the smallest changes for the estimates of $\boldsymbol{\beta}$. Furthermore, the smaller of v gives the better for robustness of parameter estimates. This suggests that tLMM, which downweights the influence of outliers and heavy-tailed noise, provides an appropriate way for achieving robust inference applied on household consumption per capita expenditure data.

5 Conclusion

This article shows the improvement in the performance of tLMM and their stability to overcome outliers compared with LMM for modeling household consumption per capita expenditure data. From the results of the analysis, we find that the best model fit to handle outliers is the tLMM with the smallest degrees of freedom (v).

The limitation of this article is the treatment of response variable which transformed to Boxcox transformation in order to obtain the symmetrical type. The next study is to develop the Generalized Linear Mixed Models (GLMM) based on the distribution of household consumption per capita expenditure data, such as the three parameters Log-normal distribution discussed in [5].

Acknowledgements. The authors would like to express a sincere gratitude to Statistics Indonesia for providing funds for this research as a part of the Ph.D. grant research program and for its human resource development projects in cooperation with Bogor Agricultural University, Indonesia.

References

[1] T. Akita and A. Pirmansyah, *Urban Inequality in Indonesia,* IUJ Research Institute, 2011.

[2] G.M. Fitzmaurice, N.M. Laird and J. Ware, *Applied Longitudinal Analysis,* Wiley, New York, 2004.

[3] D.A. Harville, Maximum likelihood approaches to variance component estimation and to related problems, *J. Am. Statist. Ass.,* **72** (1977), 320-338. https://doi.org/10.1080/01621459.1977.10480998

[4] D. Hedeker and R.D. Gibbons, *Longitudinal Data Analysis,* Wiley, Hoboken, New Jersey, 2006. https://doi.org/10.1002/0470036486

[5] P. Ismartini, N. Iriawan, Setiawan, B.S.S. Ulama, Toward a hierarchical bayesian framework for modelling the effect of regional diversity on household expenditure, *Journal of Mathematics and Statistics,* **8** (2012), 283-291. https://doi.org/10.3844/jmssp.2012.283.291

[6] A.T. James, J.T. Wiskich, R.A.J. Conyers, t-REML for robust heteroscedastic regression analysis of mitochondrial power, *Biometrics,* **49** (1993), 339-356. https://doi.org/10.2307/2532548

[7] N.M. Laird and J.H. Ware, Random effects models for longitudinal data, *Biometrics,* **38** (1982), 963-974. https://doi.org/10.2307/2529876

[8] K.L. Lange, R.J.A Little and J.M.G. Taylor, Robust statistical modelling using the t distribution, *J. Am. Statist. Ass.,* **84** (1989), 881-896. https://doi.org/10.1080/01621459.1989.10478852

[9] T.I. Lin and J.C. Lee, A robust approach to t linear mixed models applied to multiple sclerosis data, *Statist. Med.,* **25** (2006), 1397-1412. https://doi.org/10.1002/sim.2384

[10] T.I. Lin and J.C. Lee, Bayesian analysis of hierarchical linear mixed modeling using the multivariate t distribution, *J. Statist. Plann. Inference,* **137** (2007), 484-495. https://doi.org/10.1016/j.jspi.2005.12.010

[11] J.C. Pinheiro, C.H. Liu and Y.N. Wu, Efficient algorithms for robust estimation in linear mixed-effects model using the multivariate t distribution, *J. Comput. Graph. Statist.,* **10** (2001), 249-276. https://doi.org/10.1198/10618600152628059

[12] R Development Core Team, R: A Language and Environment for Statistical Computing, *R Foundation for Statistical Computing,* Vienna, 2008.

[13] SAS, *SAS/Stat Users Guide, Version 8.2. SAS Institute Inc.,* Cary, NC, 2001.

[14] P.X.K. Song, P. Zhang and A. Qu, Maximum likelihood inference in robust linear mixed-effects models using multivariate t distributions, *Statistica Sinica,* **17** (2007), 929-943.

[15] C. Sumarni, K. Sadik, K.A. Notodiputro, B. Sartono, Robustness of location estimators under t-distributions: a literature review, *IOP Conf. Series: Earth and Environmental Science,* **58** (2017), 012015. https://doi.org/10.1088/1755-1315/58/1/012015

[16] A. Ubaidillah, K.A. Notodiputro, A. Kurnia, A Fitrianto, I.W. Mangku, A robustness study of student-t distributions in regression models with application to infant birth weight data in Indonesia, *IOP Conf. Series: Earth and Environmental Science,* **58** (2017), 012013. https://doi.org/10.1088/1755-1315/58/1/012013

[17] W.L. Wang and T.H. Fan, Estimation in multivariate t linear mixed models for multiple longitudinal data, *Statistica Sinica,* **21** (2011), 1857-1880. https://doi.org/10.5705/ss.2009.306

[18] A.H. Welsh and A.M. Richardson, 13 Approaches to the robust estimation of mixed models, *Handbook of Statistics,* **15** (1997), 343-384. https://doi.org/10.1016/s0169-7161(97)15015-5

[19] M. West, Outlier models and prior distributions in Bayesian linear regression, *Journal of the Royal Statistical Society, Series B,* **46** (1984), 431-439.

Received: December 29, 2017; Published: January 15, 2018