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Estimation of Unemployment Rates Using Small Area Estimation Model by Combining Time Series and Cross-Sectional Data

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Abstract. Labor force surveys conducted over time by the rotating panel design have been carried out in many countries, including Indonesia. Labor force survey in Indonesia is regularly conducted by Statistics Indonesia (Badan Pusat Statistik-BPS) and has been known as the National Labor Force Survey (Sakernas). The main purpose of Sakernas is to obtain information about unemployment rates and its changes over time. Sakernas is a quarterly survey. The quarterly survey is designed only for estimating the parameters at the provincial level. The quarterly unemployment rate published by BPS (official statistics) is calculated based on only cross-sectional methods, despite the fact that the data is collected under rotating panel design. The study purpose to estimate a quarterly unemployment rate at the district level used small area estimation (SAE) model by combining time series and cross-sectional data. The study focused on the application and comparison between the Rao-Yu model and dynamic model in context estimating the unemployment rate based on a rotating panel survey. The goodness of fit of both models was almost similar. Both models produced an almost similar estimation and better than direct estimation, but the dynamic model was more capable than the Rao-Yu model to capture a heterogeneity across area, although it was reduced over time.

Keywords: Sakernas, rotating panel, SAE, Rao-Yu model, dynamic model.

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INTRODUCTION

Labor force surveys conducted over time by the rotating panel design has been carried out in many countries, including Indonesia. Indonesia's labor force survey has been known as the National Labor Force Survey (Sakernas). Sakernas has been regularly conducted since 1976 by Statistics Indonesia (BPS), but the rotating panel design started in 2011. The main purpose of Sakernas is to obtain information about unemployment rate and its changes over time. The quarterly survey of Sakernas is designed only for estimating the parameters at the provincial level. Estimation of parameters for the district level only conducted in 3-rd quarter in each year. It requires 3 times more than the number of samples for each quarter which implies the requirement of additional cost and time. Also, the official quarterly unemployment rates is estimated based on only cross-sectional methods despite the fact that the data are collected under rotating panel design. There are two issues, that is insufficient sample size and the gaps in the parameter estimation. To solve these issues, it is necessary to study how to estimate parameter for the district level based on rotating panel survey when sample size is insufficient.

Small area estimation (SAE) model is one of an alternative to estimate parameter for small area. In the context of SAE, an area or domain becomes small when its sample size too small for direct estimation of adequate precision [6]. SAE model is a mixed model, the fixed effects come from auxiliary information and random effects come from the specific variety of the area. There are two ways to improve estimation, that is borrow strength over area and/or borrow strength over time. One model that borrows both the strength is the Rao-Yu model which developed by Rao and Yu

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[8,9]. The Rao-Yu model used for a combination of time series and cross-sectional data. The model is an extension of the basic Fay-Herriot model [12] by adding a random area-time component which follows an autoregressive process order 1 or AR (1).

The Rao-Yu model is not specifically developed for panel rotation data, so have not facilitated the possibility of the presence of bias that caused by the rotation. Boonstra developed a multi-level time-series models for small area estimation based on the data panel rotation [2]. The models add a components of bias due to the rotation. The model was applied to estimate a quarterly unemployment rate based on the Dutch labor force survey. The survey used a rotation panel design.The Dutch LFS uses a rotating panel design which in each quarter the total sample consists of five groups. The rotation group biases are handled in the model by including measurement effects for the second to fifth waves.

A complicated issue in the Rao-Yu model is that estimation of parameters is subject to a constraints, especially the constraints of autoregressive coefficient. Some researchers have been modify the model related to the constraints of autoregressive coefficient. Datta, Lahiri and Maiti have used restricted maximum likelihood estimation (REML) under the random walk model [7]. The complicating issues in parameters estimation for the Rao-Yu model have been resolved by Fay and Diallo [5] and Fay, Planty and Diallo [4]. They used the maximum likelihood (ML) estimation and restricted maximum likelihood (REML) estimation and they have successfully implemented an iterative Newton-Raphson algorithm in R software. Also, they have modified the Rao-Yu model by removing the stationarity requirement and modifying the random effects. They call the model as a dynamic model. Furthermore, Diallo also has been generalized the Rao-Yu model [3].

This is a preliminary study related to the issues of insufficiency of sample size in the rotating panel design, especially in the case of Sakernas. The study have not yet considered the possibility of the presence of bias caused by the rotation group. The study purpose to estimate a quarterly unemployment rate at the district level using a SAE model by combining time series and cross-sectional data. The study focused on the application and comparison between the Rao-Yu model and the dynamic model. The data used was Sakernas data of West Java Province from February 2011 till February 2014 (13 quarters). The study was structured as follows. Section 2 briefly described about Sakernas, section 3 described about the Rao-Yu model and the dynamic model, and the EBLUP and MSE estimation of them which based on ML and REML methods, section 4 described a results and section 5 discussed about the results.

SAKERNAS

Started in 2011, Sakernas was designed based on three stages sampling design. First, select some primary sampling unit (PSU) by probability proportional to size (PPS) sampling. Second, select a census block by PPS sampling from the each PSU that was selected at the first stage. Third, select some household by linear systematic sampling from the each census block that was selected at the second stage. The total sample for West Java province was 400 census blocks (or 4000 households) for each quarter. The sample was allocated proportionally to 26 districts. The total sample for each district is about 8-21 census blocks (or 80-210 households) for each quarter. The sample was only about 0.3 percent of population (133,162 census blocks).

Sakernas is household survey conducted according to a rotating panel design. in the framework of the rotation panel, the total sample of census blocks are separated into four sample package (1,2,3,4). Every package, at each census block is formed 4 household groups. The groups at package-1 is A, E, I and M, package-2 is B, F, J and N, package-3 is C, G, K and O, and package-4 is D, H, L and P. So that in every quarter the total sample consists of four groups coming from different packages. Each group in the census block consists of about 10 households. Sakernas rotation panel designed on quarterly basis by maintaining 3/4 groups of the previous quarter and add 1/4 of new groups of the current quarter. FIGURE 1 illustrates the rotating panel design for February 2011 till February 2014. The same colour identifies that the groups derived from the same census block. [1]

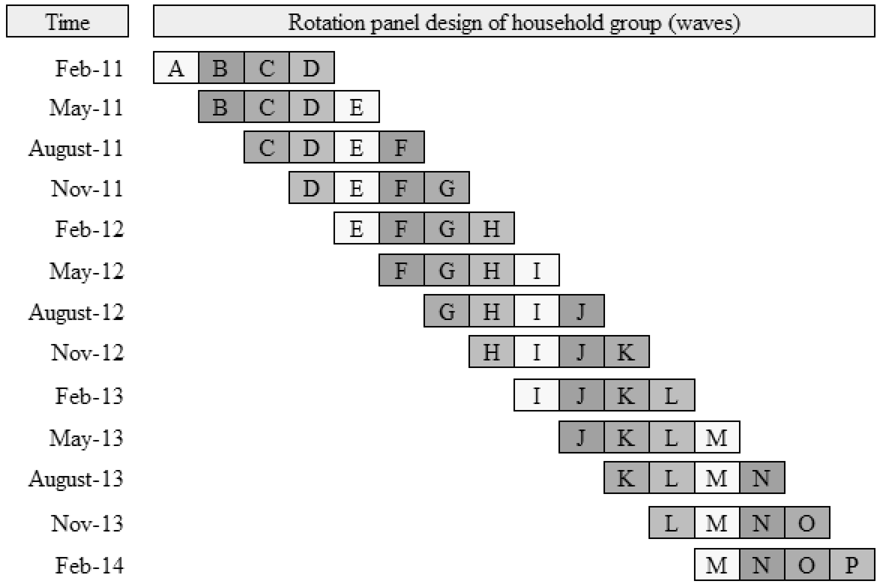


FIGURE 1. Sakernas rotating panel design.

THE RAO-YU MODEL AND DYNAMIC MODEL

Let y_{it} be a direct survey estimator of i-th small area mean at time point t, say θ_{it}. The y_{it} is assumed unbiased for θ_{it}. The θ_{it}'s are related to $y_{it} = \theta_{it} + e_{it}$, where the e_{it}'s are sampling errors that normally distributed with $E(e_{it}) = 0$ and known block diagonal covariance matrix $\mathbf{\Sigma}$. A vector of auxiliary variables, $\mathbf{x}_{it}$ related to θ_{it} is available such that $\theta_{it} = \mathbf{x}'_{it}\boldsymbol{\beta} + v_i + u_{it}$, where $\boldsymbol{\beta}$ is a vector of regression coefficients, v_i is a random effect for small area i and u_{it} is a random effect for small area i at time point t. The Rao-Yu model is defined as follows

$$y_{it} = \mathbf{x}'_{it}\boldsymbol{\beta} + v_i + u_{it} + e_{it}, \quad i = 1,2,\ldots,m, \quad t = 1,2,\ldots,T, \tag{1}$$

where $\mathbf{x}'_{it} = \left(x_{it1},\ldots,x_{itp}\right)'$ is a vector of p fixed auxiliary variables for small area i at time t and $v_i \sim iidN(0,\sigma_v^2)$. The u_{it}'s are assumed to follow a first order autoregressive process within each area i,

$$u_{it} = \rho u_{i,t-1} + \varepsilon_{it}, \ |\rho| < 1, \tag{2}$$

with $\varepsilon_{it} \sim iidN(0,\sigma^2)$. The errors v_i, ε_{it} and e_{it} are assumed to be independent of each other. The condition $|\rho| < 1$ ensures stationarity of the series defined by (2) in order to obtain an autoregressive process of order 1. The stationarity assumption leads to:

$$Var(u_{it}) = \frac{\sigma^2}{1-\rho^2} \tag{3}$$

Fay and Diallo have modified the Rao-Yu model by removing the stationarity requirement and modifying the random effect terms [5]. They call the model as dynamic model. Unlike the Rao-Yu model, which assumes stationarity (2) for $|\rho| < 1$, the dynamic model does not assume stationarity, ρ is not constrained. The model is defined as follows

$$y_{it} = \mathbf{x}'_{it}\boldsymbol{\beta} + \rho^{t-1}v_i^* + u_{it}^* + e_{it}, \ \ i = 1,2,\ldots,m, \quad t = 1,2,\ldots,T, \tag{4}$$

where v_i^* is a random effect for small area i at $t=1$, $v_i^* \sim iidN(0, \sigma_{v^*}^2)$, $u_{i1}^* = 1$, $u_{it}^* = \rho u_{it}^* + \varepsilon_{it}$, for $t > 1$ with $\varepsilon_{it} \sim iidN(0, \sigma^2)$. When $\rho > 1$, the model corresponds to a divergent situation in which areas become progressively more disparate. But by dropping the stationarity assumption, the dynamic model is more appropriate for a situation in which the disparity among area dissipates over time. When $\sigma_{v^*}^2 = \frac{\sigma^2}{1-\rho^2}$ and $|\rho| < 1$, the dynamic model becomes equivalent a Rao-Yu model with $\sigma_v^2 = 0$ [4].

The Rao-Yu and the dynamic model are a special case of the general linear mixed. Both models cover many small area models and the covariance matrices have block diagonal form when arranged by area. Arranging the data $\{y_{it}\}$ as $\mathbf{y} = (y_{11}, \dots, y_{1T}; \dots; y_{m1}, \dots, y_{mT})' = (\mathbf{y}_1', \dots, \mathbf{y}_m')'$, for each area-$i$, the Rao-Yu model (1) may be written, in matrix form, as

$$\mathbf{y}_i = \mathbf{X}_i \boldsymbol{\beta} + \mathbf{Z}_i \mathbf{v}_i + \mathbf{e}_i, \tag{5}$$

with

$$\begin{aligned} &\mathbf{X}_i = (\mathbf{x}_{i1}, \dots, \mathbf{x}_{iT})', && \boldsymbol{\beta} = (\beta_1, \dots, \beta_p)', && \mathbf{Z}_i = (\mathbf{1}_T, \mathbf{I}_T), \\ &\mathbf{v}_i = (v_i, \mathbf{u}_i')', && \mathbf{u}_i = (u_{i1}, \dots, u_{iT})', && \mathbf{e}_i = (e_{i1}, \dots, e_{iT})', \end{aligned} \tag{6}$$

where $\mathbf{I}_T$ is the identity matrix of order T and $\mathbf{1}_T$ is a T-vector of 1's.

The e_{it}'s are sampling errors that normally distributed with $E(e_{it}) = 0$ and known block diagonal covariance matrix $\boldsymbol{\Sigma}$, $v_i \sim iidN(0, \sigma_v^2)$, $Var(u_{it}) = \frac{\sigma^2}{1-\rho^2}$, and v_i and $\mathbf{u}_i$ are independent, so that

$$\begin{aligned} &E(\mathbf{v}_i) = \mathbf{0}, && Cov(\mathbf{v}_i) = Cov(v_i, \mathbf{u}_i) = \mathbf{G}_i = \begin{pmatrix} \sigma_v^2 & \mathbf{o}_T' \\ \mathbf{o}_T & \sigma^2 \boldsymbol{\Gamma} \end{pmatrix} \\ &E(\mathbf{e}_i) = \mathbf{0}, && Cov(\mathbf{e}_i) = \boldsymbol{\Sigma}_i \end{aligned} \tag{7}$$

where $\mathbf{o}_T$ is a T-vector of 0's and $\sigma^2 \boldsymbol{\Gamma}$ is the $T \times T$ covariance matrix of $\mathbf{u}_i$. $\boldsymbol{\Gamma}$ is a symmetric matrix with elements $\rho^{|j-k|}/(1-\rho^2)$. The v_i, $\mathbf{u}_i$ and $\mathbf{e}_i$ are mutually independent, so that the covariance matrix of $\mathbf{y}_i$ is

$$\begin{aligned} Cov(\mathbf{y}_i) = \mathbf{V}_i &= Cov(\mathbf{Z}_i \mathbf{v}_i + \mathbf{e}_i) \\ &= Cov(\mathbf{Z}_i \mathbf{v}_i) + Cov(\mathbf{e}_i) \\ &= \mathbf{Z}_i Cov(\mathbf{v}_i) \mathbf{Z}_i' + \boldsymbol{\Sigma}_i \\ &= \mathbf{Z}_i \mathbf{G}_i \mathbf{Z}_i' + \boldsymbol{\Sigma}_i \\ &= (\mathbf{1}_T, \mathbf{I}_T) \begin{pmatrix} \sigma_v^2 & \mathbf{o}_T' \\ \mathbf{o}_T & \sigma^2 \boldsymbol{\Gamma} \end{pmatrix} (\mathbf{1}_T, \mathbf{I}_T)' + \boldsymbol{\Sigma}_i \\ &= \sigma_v^2 \mathbf{J}_T + \sigma^2 \boldsymbol{\Gamma} + \boldsymbol{\Sigma}_i \end{aligned} \tag{8}$$

where $\mathbf{J}_T = \mathbf{1}_T \mathbf{1}_T'$ is $T \times T$ matrix with elements 1. Under the Rao-Yu model, for all area, model (1) may be written in matrix form, as

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{Z}\mathbf{v} + \mathbf{e}, \tag{9}$$

with

$$\begin{aligned} &\mathbf{X} = (\mathbf{X}_1', \dots, \mathbf{X}_m')', && \mathbf{Z} = \mathbf{I}_m \otimes \mathbf{Z}_i \\ &\mathbf{v} = (\mathbf{v}_1', \dots, \mathbf{v}_m')', && \mathbf{e} = (\mathbf{e}_1', \dots, \mathbf{e}_m')', \end{aligned}$$

and the covariance matrix of $\mathbf{y}$ has a block diagonal structure, such that

$$
\begin{aligned}
Cov(\mathbf{y}) = \mathbf{V} &= \text{blockdiag}_i(\mathbf{V}_i) \\
&= \text{blockdiag}_i(\mathbf{Z}_i\mathbf{G}_i\mathbf{Z}_i' + \boldsymbol{\Sigma}_i) \\
&= \text{blockdiag}_i(\sigma_v^2\mathbf{J}_T + \sigma^2\boldsymbol{\Gamma} + \boldsymbol{\Sigma}_i) \\
&= \mathbf{ZGZ}' + \boldsymbol{\Sigma} \qquad (10)
\end{aligned}
$$

with $\mathbf{G} = \mathbf{I}_m \otimes \mathbf{G}_i$ and $\boldsymbol{\Sigma} = \text{blockdiag}_i(\boldsymbol{\Sigma}_i)$.

Under the dynamic model, for each area-i, model (4) may be written, in matrix form, as

$$
\mathbf{y}_i = \mathbf{X}_i\boldsymbol{\beta} + \mathbf{Z}_i^*\mathbf{v}_i^* + \mathbf{e}_i, \qquad (11)
$$

with

$$
\mathbf{Z}_i^* = (\boldsymbol{\rho}_T, \mathbf{I}_T), \quad E(\mathbf{v}_i^*) = E(v_i^*, \mathbf{u}_i^*) = \mathbf{0}, \quad Cov(v_i^*, \mathbf{u}_i^*) = \mathbf{G}_i^* = \begin{pmatrix} \sigma_{v^*}^2 & \mathbf{o}_T' \\ \mathbf{o}_T & \sigma^2\boldsymbol{\Gamma}_{u^*} \end{pmatrix} \qquad (12)
$$

where $\boldsymbol{\rho}_T$ is a T-vector with elements $\rho^{(j-1)}$, where $j = 1, \dots, T$, and $\boldsymbol{\Gamma}_{u^*}$ is the $T \times T$ covariance matrix of $\mathbf{u}_i^*$. $\boldsymbol{\Gamma}_{u^*}$ is a symmetric matrix with elements $\boldsymbol{\Gamma}_{u^*(1,k)} = 0$ and $\boldsymbol{\Gamma}_{u^*(j,k)} = \rho^{(k-j)} \sum_{j'=1}^{j-1} \rho^{(2j'-2)}$ for $1 < j \leq k$. The $v_i^*, \mathbf{u}_i^*$ and $\mathbf{e}_i$ are mutually independent, so that under the dynamic model, the covariance matrix of $\mathbf{y}_i$ is

$$
\begin{aligned}
Cov(\mathbf{y}_i) = \mathbf{V}_i^* &= Cov(\mathbf{Z}_i^*\mathbf{v}_i^* + \mathbf{e}_i) \\
&= \mathbf{Z}_i^*\mathbf{G}_i^*\mathbf{Z}_i^{*\prime} + \boldsymbol{\Sigma}_i \\
&= (\boldsymbol{\rho}_T, \mathbf{I}_T)\begin{pmatrix} \sigma_{v^*}^2 & \mathbf{o}_T' \\ \mathbf{o}_T & \sigma^2\boldsymbol{\Gamma}_{u^*} \end{pmatrix}(\boldsymbol{\rho}_T, \mathbf{I}_T)' + \boldsymbol{\Sigma}_i \\
&= \sigma_{v^*}^2\boldsymbol{\Gamma}_{v^*} + \sigma^2\boldsymbol{\Gamma}_{u^*} + \boldsymbol{\Sigma}_i \qquad (13)
\end{aligned}
$$

where $\boldsymbol{\Gamma}_{v^*} = \boldsymbol{\rho}_T\boldsymbol{\rho}_T'$ is a $T \times T$ symmetric matrix with elements $\rho^{(j+k-2)}$, with j is row and k is column, $j = 1, \dots, T$ and $k = 1, \dots, T$. For all area, model (4) may be written, in matrix form, as

$$
\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{Z}^*\mathbf{v}^* + \mathbf{e}, \qquad (14)
$$

with

$$
\begin{aligned}
\mathbf{X} &= (\mathbf{X}_1', \dots, \mathbf{X}_m')', & \mathbf{Z}^* &= \mathbf{I}_m \otimes \mathbf{Z}_i^* \\
\mathbf{v}^* &= \left(\mathbf{v}_i^{*\prime}, \dots, \mathbf{v}_m^{*\prime}\right)', & \mathbf{e} &= (\mathbf{e}_1', \dots, \mathbf{e}_m')',
\end{aligned}
$$

and the covariance matrix of $\mathbf{y}$ has a block diagonal structure, such that

$$
\begin{aligned}
Cov(\mathbf{y}) = \mathbf{V}^* &= \text{blockdiag}_i(\mathbf{V}_i^*) \\
&= \text{blockdiag}_i\left(\mathbf{Z}_i^*\mathbf{G}_i^*\mathbf{Z}_i^{*\prime} + \boldsymbol{\Sigma}_i\right) \\
&= \text{blockdiag}_i\left(\sigma_{v^*}^2\boldsymbol{\Gamma}_{v^*} + \sigma^2\boldsymbol{\Gamma}_{u^*} + \boldsymbol{\Sigma}_i\right) \\
&= \mathbf{Z}^*\mathbf{G}^*\mathbf{Z}^{*\prime} + \boldsymbol{\Sigma} \qquad (15)
\end{aligned}
$$

with $\mathbf{G}^* = \mathbf{I}_m \otimes \mathbf{G}_i^*$ and $\boldsymbol{\Sigma} = \text{blockdiag}_i(\boldsymbol{\Sigma}_i)$.

The BLUP and the EBLUP

A general linear mixed model, $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{Z}\mathbf{v} + \mathbf{e}$, where $\mathbf{y}$ is an $n \times 1$ observation vector, $\mathbf{X}$ is a known, $n \times p$ matrix of full rank, $\boldsymbol{\beta}$ is an unknown, fixed vector, and $\mathbf{Z}$ is a known, $n \times q$ matrix of full rank. $\mathbf{v}$ and $\mathbf{e}$ are non-observable random vectors with $E(\mathbf{v}) = \mathbf{0}$, $E(\mathbf{e}) = \mathbf{0}$ and $Var\begin{pmatrix}\mathbf{v}\\ \mathbf{e}\end{pmatrix} = \sigma^2\begin{pmatrix}\mathbf{G} & 0\\ 0 & \boldsymbol{\Sigma}\end{pmatrix}$, with σ^2 is a scalar, and $\mathbf{G}$ and $\boldsymbol{\Sigma}$ are both non-singular. Henderson [13] obtained the general form of BLUP (Best Linear Unbiased Prediction) for the general linear mixed model is

$$\hat{y}_{i_{BLUP}} = \mathbf{x}_i\tilde{\boldsymbol{\beta}} + \mathbf{k}'\mathbf{G}\mathbf{Z}'\mathbf{V}^{-1}(\mathbf{y} - \mathbf{X}\tilde{\boldsymbol{\beta}}), \tag{16}$$

where $\mathbf{k}$ is the n-vector with 1 in the i-th position and 0 else where, $\mathbf{V} = \mathbf{Z}\mathbf{G}\mathbf{Z}' + \boldsymbol{\Sigma}$, and $\tilde{\boldsymbol{\beta}} = (\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1}(\mathbf{X}'\mathbf{V}^{-1}\mathbf{y})$ is the generalized least squares estimator of $\boldsymbol{\beta}$.

For the Rao-Yu and dynamic model, $n = T \times m$. Because the covariance matrix has block diagonal form when arranged by area, so that under the Rao-Yu model (5), with $\mathbf{Z}_i$, $\mathbf{G}_i$ and $\mathbf{V}_i$ are defined at (6), (7) and (8), respectively, based on (16), assuming that σ_v^2, σ^2 and ρ are known, the BLUP for the current occasion T may be expressed as

$$\hat{y}_{iT_{BLUPRY}} = \mathbf{x}_{iT}\tilde{\boldsymbol{\beta}} + \mathbf{k}_i'\mathbf{G}_i\mathbf{Z}_i'\mathbf{V}_i^{-1}(\mathbf{y}_i - \mathbf{X}_i\tilde{\boldsymbol{\beta}}), \tag{17}$$

with $\mathbf{k}_i = (1, \mathbf{0}_{T-1}', 1)'$, where $\mathbf{0}_{T-1}$ is a $(T-1)$-vector with elements 0. Further,

$$\begin{aligned}
\hat{y}_{iT_{BLUPRY}} &= \mathbf{x}_{iT}'\tilde{\boldsymbol{\beta}} + (1, \mathbf{0}_{T-1}', 1)\begin{pmatrix}\sigma_v^2 & \mathbf{o}_T'\\ \mathbf{o}_T & \sigma^2\boldsymbol{\Gamma}\end{pmatrix}(\mathbf{1}_T, \mathbf{I}_T)'(\sigma_v^2\mathbf{J}_T + \sigma^2\boldsymbol{\Gamma} + \boldsymbol{\Sigma}_i)^{-1}(\mathbf{y}_i - \mathbf{X}_i\tilde{\boldsymbol{\beta}})\\
&= \mathbf{x}_{iT}'\tilde{\boldsymbol{\beta}} + (\sigma_v^2, \sigma^2\boldsymbol{\gamma}_T')(\mathbf{1}_T, \mathbf{I}_T)'(\sigma_v^2\mathbf{J}_T + \sigma^2\boldsymbol{\Gamma} + \boldsymbol{\Sigma}_i)^{-1}(\mathbf{y}_i - \mathbf{X}_i\tilde{\boldsymbol{\beta}})\\
&= \mathbf{x}_{iT}'\tilde{\boldsymbol{\beta}} + (\sigma_v^2\mathbf{1}_T + \sigma^2\boldsymbol{\gamma}_T)'(\sigma_v^2\mathbf{J}_T + \sigma^2\boldsymbol{\Gamma} + \boldsymbol{\Sigma}_i)^{-1}(\mathbf{y}_i - \mathbf{X}_i\tilde{\boldsymbol{\beta}}),
\end{aligned} \tag{18}$$

where $\boldsymbol{\gamma}_T'$ is the T-th row of $\boldsymbol{\Gamma}$ or $\boldsymbol{\gamma}_T$ is the T-th column of $\boldsymbol{\Gamma}$, because $\boldsymbol{\Gamma}$ is a symmetric matrix.

Under the dynamic model (11), with $\mathbf{Z}_i^*$ and $\mathbf{G}_i^*$ are defined at (12), and $\mathbf{V}_i^*$ are defined at (13), based on (16), assuming that $\sigma_{v^*}^2$, σ^2 and ρ are known, the BLUP for the current occasion T may be expressed as

$$\hat{y}_{iT_{BLUPDYN}} = \mathbf{x}_{iT}\tilde{\boldsymbol{\beta}} + \mathbf{k}_i^{*\prime}\mathbf{G}_i^*\mathbf{Z}_i^{*\prime}\mathbf{V}_i^{*-1}(\mathbf{y}_i - \mathbf{X}_i\tilde{\boldsymbol{\beta}}), \tag{19}$$

with $\mathbf{k}_i^* = (\rho^{T-1}, \mathbf{0}_{T-1}', 1)'$. Further,

$$\begin{aligned}
\hat{y}_{iT_{BLUPDYN}} &= \mathbf{x}_{iT}'\tilde{\boldsymbol{\beta}} + (\rho^{T-1}, \mathbf{0}_{T-1}', 1)\begin{pmatrix}\sigma_{v^*}^2 & \mathbf{o}_T'\\ \mathbf{o}_T & \sigma^2\boldsymbol{\Gamma}_{u^*}\end{pmatrix}(\boldsymbol{\rho}_T, \mathbf{I}_T)'\left(\sigma_{v^*}^2\boldsymbol{\Gamma}_{v^*} + \sigma^2\boldsymbol{\Gamma}_{u^*} + \boldsymbol{\Sigma}_i\right)^{-1}(\mathbf{y}_i - \mathbf{X}_i\tilde{\boldsymbol{\beta}})\\
&= \mathbf{x}_{iT}'\tilde{\boldsymbol{\beta}} + \left(\sigma_{v^*}^2\rho^{T-1}, \sigma^2\boldsymbol{\gamma}_{u^*,T}'\right)(\boldsymbol{\rho}_T, \mathbf{I}_T)'\left(\sigma_{v^*}^2\boldsymbol{\Gamma}_{v^*} + \sigma^2\boldsymbol{\Gamma}_{u^*} + \boldsymbol{\Sigma}_i\right)^{-1}(\mathbf{y}_i - \mathbf{X}_i\tilde{\boldsymbol{\beta}})\\
&= \mathbf{x}_{iT}'\tilde{\boldsymbol{\beta}} + \left(\sigma_{v^*}^2\boldsymbol{\gamma}_{v^*,T}' + \sigma^2\boldsymbol{\gamma}_{u^*,T}'\right)\left(\sigma_{v^*}^2\boldsymbol{\Gamma}_{v^*} + \sigma^2\boldsymbol{\Gamma}_{u^*} + \boldsymbol{\Sigma}_i\right)^{-1}(\mathbf{y}_i - \mathbf{X}_i\tilde{\boldsymbol{\beta}})\\
&= \mathbf{x}_{iT}'\tilde{\boldsymbol{\beta}} + \left(\sigma_{v^*}^2\boldsymbol{\gamma}_{v^*,T} + \sigma^2\boldsymbol{\gamma}_{u^*,T}\right)'\left(\sigma_{v^*}^2\boldsymbol{\Gamma}_{v^*} + \sigma^2\boldsymbol{\Gamma}_{u^*} + \boldsymbol{\Sigma}_i\right)^{-1}(\mathbf{y}_i - \mathbf{X}_i\tilde{\boldsymbol{\beta}}),
\end{aligned} \tag{20}$$

where $\boldsymbol{\gamma}_{v^*,T}'$ is the T-th row of $\boldsymbol{\Gamma}_{v^*}$ or $\boldsymbol{\gamma}_{v^*,T}$ is the T-th column of $\boldsymbol{\Gamma}_{v^*}$ and $\boldsymbol{\gamma}_{u^*,T}'$ is the T-th row of $\boldsymbol{\Gamma}_{u^*}$ or $\boldsymbol{\gamma}_{u^*,T}$ is the T-th column of $\boldsymbol{\Gamma}_{u^*}$.

In fact, the parameters σ_v^2, σ^2, $\sigma_{v^*}^2$ and ρ are unknown then estimated by $\hat{\sigma}_v^2$, $\hat{\sigma}^2$, $\hat{\sigma}_{v^*}^2$ and $\hat{\rho}$, respectively. By replacing the unknown parameters by their estimators, will be obtained the EBLUP (Empirical Best Linear Unbiased Prediction), that is

$$\hat{y}_{iT_{EBLUPRY}} = \mathbf{x}_{iT}'\tilde{\boldsymbol{\beta}} + (\hat{\sigma}_v^2\mathbf{1}_T + \hat{\sigma}^2\hat{\boldsymbol{\gamma}}_T)'\left(\hat{\sigma}_v^2\mathbf{J}_T + \hat{\sigma}^2\hat{\boldsymbol{\Gamma}} + \boldsymbol{\Sigma}_i\right)^{-1}(\mathbf{y}_i - \mathbf{X}_i\tilde{\boldsymbol{\beta}}), \tag{21}$$

and

$$\hat{y}_{iT_{EBLUPDYN}} = \mathbf{x}'_{iT}\widetilde{\boldsymbol{\beta}} + \left(\hat{\sigma}^2_{v^*}\hat{\boldsymbol{\gamma}}_{v^*,T} + \hat{\sigma}^2\hat{\boldsymbol{\gamma}}_{u^*,T}\right)'\left(\hat{\sigma}^2_{v^*}\hat{\boldsymbol{\Gamma}}_{v^*} + \hat{\sigma}^2\hat{\boldsymbol{\Gamma}}_{u^*} + \boldsymbol{\Sigma}_i\right)^{-1}\left(\mathbf{y}_i - \mathbf{X}_i\widetilde{\boldsymbol{\beta}}\right) \tag{22}$$

where $\hat{y}_{iT_{EBLUPRY}}$ is the EBLUP under the Rao-Yu model (5) and $\hat{y}_{iT_{EBLUPDYN}}$ is the EBLUP under the dynamic model (11).

Rao and Yu [8,9] used two approaches to estimate the parameters, i.e. by assuming ρ is known and ρ is unknown. For the case of an AR (1) model with known ρ, they used two ordinary least squares steps to obtain unbiased estimators $\hat{\sigma}^2_v$ and $\hat{\sigma}^2$, which are then truncated to zero. The truncated estimators $\hat{\sigma}^2_v$ and $\hat{\sigma}^2$ are not unbiased, but they are consistent as $m \to \infty$. For ρ is unknown, Rao and Yu [8,9] used two methods to estimate ρ, i.e. moment estimator and naive estimator. They obtained a consistent moment estimator $\hat{\rho}$, but it often takes value outside the admissible range (-1,1), especially for small T or small σ^2 relative to the sampling variation. They have obtained that the naive estimator is inconsistent in the presence of sampling error and typically under-estimate ρ.

Fay and Diallo [5] used ML and REML estimation to estimate the parameters for the Rao-Yu model. They used an iterative Newton-Raphson algorithm and they have successfully implemented the algorithm in R software. The logarithm of the likelihood function associated with the model (9) is equal to

$$ln\,L\left(\boldsymbol{\beta}, \mathbf{V}_{(\Omega)}\right) = -\tfrac{mT}{2}ln(2\pi) - \tfrac{1}{2}ln\left|\mathbf{V}_{(\Omega)}\right| - \tfrac{1}{2}(\mathbf{y} - \mathbf{X}\boldsymbol{\beta})'\mathbf{V}^{-1}_{(\Omega)}(\mathbf{y} - \mathbf{X}\boldsymbol{\beta}). \tag{23}$$

Notice that $-\frac{mT}{2}ln(2\pi)$ is a constant that does not depend on the unknown variance component in the model, and therefore, can be ignored. So that equation (23) become

$$ln\,L\left(\boldsymbol{\beta}, \mathbf{V}_{(\Omega)}\right) = -\tfrac{1}{2}ln\left|\mathbf{V}_{(\Omega)}\right| - \tfrac{1}{2}(\mathbf{y} - \mathbf{X}\boldsymbol{\beta})'\mathbf{V}^{-1}_{(\Omega)}(\mathbf{y} - \mathbf{X}\boldsymbol{\beta}). \tag{24}$$

Under the Rao-Yu model, $\mathbf{V}_{(\Omega)} = \mathbf{V}$ is defined as (10) with $\Omega = (\sigma^2_v, \sigma^2, \rho)$ and under the dynamic model, $\mathbf{V}_{(\Omega)} = \mathbf{V}^*$ is defined as (15) with $\Omega = \left(\sigma^2_{v^*}, \sigma^2, \rho\right)$.

The maximum likelihood estimated that maximize the log likelihood function (24) over the valid range of parameter values are obtained by resolving $\frac{\partial \ln L\left(\boldsymbol{\beta}, \mathbf{V}_{(\Omega)}\right)}{\partial \Omega_j} = 0$, where

$$\begin{aligned}
\frac{\partial\, ln\,L\left(\boldsymbol{\beta}, \mathbf{V}_{(\Omega)}\right)}{\partial \Omega_j} &= \frac{\partial\left(-\frac{1}{2}ln\left|\mathbf{V}_{(\Omega)}\right| - \frac{1}{2}(\mathbf{y} - \mathbf{X}\boldsymbol{\beta})'\mathbf{V}^{-1}_{(\Omega)}(\mathbf{y} - \mathbf{X}\boldsymbol{\beta})\right)}{\partial \Omega_j} \\
&= -\frac{1}{2}\frac{\partial\left(ln\left|\mathbf{V}_{(\Omega)}\right|\right)}{\partial \Omega_j} - \frac{1}{2}(\mathbf{y} - \mathbf{X}\boldsymbol{\beta})'\frac{\partial\left(\mathbf{V}^{-1}_{(\Omega)}\right)}{\partial \Omega_j}(\mathbf{y} - \mathbf{X}\boldsymbol{\beta}) \\
&= -\frac{1}{2}tr\left(\mathbf{V}^{-1}_{(\Omega)}\frac{\partial\left(\mathbf{V}_{(\Omega)}\right)}{\partial \Omega_j}\right) + \frac{1}{2}(\mathbf{y} - \mathbf{X}\boldsymbol{\beta})'\left(\mathbf{V}^{-1}_{(\Omega)}\frac{\partial\left(\mathbf{V}_{(\Omega)}\right)}{\partial \Omega_j}\mathbf{V}^{-1}_{(\Omega)}\right)(\mathbf{y} - \mathbf{X}\boldsymbol{\beta}) \\
&= -\frac{1}{2}tr\left(\mathbf{V}^{-1}_{(\Omega)}\mathbf{V}_{(\Omega_j)}\right) + \frac{1}{2}(\mathbf{y} - \mathbf{X}\boldsymbol{\beta})'\mathbf{V}^{-1}_{(\Omega)}\mathbf{V}_{(\Omega_j)}\mathbf{V}^{-1}_{(\Omega)}(\mathbf{y} - \mathbf{X}\boldsymbol{\beta})
\end{aligned} \tag{25}$$

with $\mathbf{V}_{(\Omega_j)} = \frac{\partial \mathbf{V}_{(\Omega)}}{\partial \Omega_j}$. Next resolve equation (25) with $\frac{\partial \ln L\left(\boldsymbol{\beta}, \mathbf{V}_{(\Omega)}\right)}{\partial \Omega_j} = 0$, result

$$\frac{1}{2}tr\left(\mathbf{V}^{-1}_{(\Omega)}\mathbf{V}_{(\Omega_j)}\right) = \frac{1}{2}(\mathbf{y} - \mathbf{X}\boldsymbol{\beta})'\mathbf{V}^{-1}_{(\Omega)}\mathbf{V}_{(\Omega_j)}\mathbf{V}^{-1}_{(\Omega)}(\mathbf{y} - \mathbf{X}\boldsymbol{\beta}) \tag{26}$$

The equation (26) is solved simultaneously and often iteratively. The Fisher scoring method, a variation of the Newton-Raphson method, for solving the equation is equal at step $(s + 1)$ to

$$\Omega^{(s+1)} = \Omega^{(s)} + \left(\boldsymbol{\Psi}_{ML(\Omega^{(s)})}\right)^{-1} \frac{\partial\, ln\, L\left(\widetilde{\boldsymbol{\beta}}_{(\Omega^{(s)})}, \Omega^{(s)}\right)}{\partial\Omega} \tag{27}$$

and

$$\boldsymbol{\beta}^{(s+1)} = \widetilde{\boldsymbol{\beta}}_{(\Omega^{(s+1)})}.$$

where $\boldsymbol{\Psi}_{ML(\Omega)}$ is a 3×3 ML information matrix associated with Ω. The (j, k)-th element of $\boldsymbol{\Psi}_{ML(\Omega)}$ defined by

$$\begin{aligned}\left(\boldsymbol{\Psi}_{ML(\Omega)}\right)_{jk} &= -E\left(\frac{\partial^2\, ln\, L\left(\boldsymbol{\beta}, \mathbf{V}_{(\Omega)}\right)}{\partial\Omega_j \partial\Omega_k}\right) \\ &= \frac{1}{2} tr\left(\mathbf{V}_{(\Omega)}^{-1}\mathbf{V}_{(\Omega_j)}\mathbf{V}_{(\Omega)}^{-1}\mathbf{V}_{(\Omega_k)}\right).\end{aligned} \tag{28}$$

At the convergence of the Fisher scoring iterations (27), will be get the ML estimator $\widehat{\Omega}_{ML}$ of Ω and $\widehat{\boldsymbol{\beta}}_{ML} = \widetilde{\boldsymbol{\beta}}_{(\widehat{\Omega}_{ML})}$ of $\boldsymbol{\beta}$. The asymptotic covariance matrix of $\widehat{\boldsymbol{\beta}}_{ML}$ and $\widehat{\Omega}_{ML}$ has a block diagonal structure, $diag\left(\overline{\mathbf{V}}\left(\widehat{\boldsymbol{\beta}}_{ML}\right), \overline{\mathbf{V}}\left(\widehat{\Omega}_{ML}\right)\right)$, with $\overline{\mathbf{V}}\left(\widehat{\boldsymbol{\beta}}_{ML}\right) = \left(\mathbf{X}'\mathbf{V}_{(\Omega)}^{-1}\mathbf{X}\right)^{-1}$ is covariance matrix of $\widehat{\boldsymbol{\beta}}_{ML}$ and $\overline{\mathbf{V}}\left(\widehat{\Omega}_{ML}\right) = \left(\boldsymbol{\Psi}_{ML(\Omega)}\right)^{-1}$ is covariance matrix of $\widehat{\Omega}_{ML}$.

REML estimation is a modification of ML procedur. In REML, however, the likelihood function of a set of error contrasts is used, denoted by $\mathbf{c}'\mathbf{y}$, where $\mathbf{c}'\mathbf{X} = \mathbf{0}$, and $\mathbf{c}$ has rank equal to $mT - r(\mathbf{X})$. Under REML estimation the log likelihood function is

$$\begin{aligned} ln\, L\left(\mathbf{V}_{(\Omega)}\right) &= -\left(\frac{mT - r(\mathbf{X})}{2}\right) ln(2\pi) - \frac{1}{2} ln\left|\mathbf{c}'\mathbf{V}_{(\Omega)}\mathbf{c}\right| - \frac{1}{2}(\mathbf{c}'\mathbf{y} - \mathbf{c}'\mathbf{X}\boldsymbol{\beta})'\left(\mathbf{c}'\mathbf{V}_{(\Omega)}\mathbf{c}\right)^{-1}(\mathbf{c}'\mathbf{y} - \mathbf{c}'\mathbf{X}\boldsymbol{\beta}) \\ &= -\left(\frac{mT - r(\mathbf{X})}{2}\right) ln(2\pi) - \frac{1}{2} ln\left|\mathbf{c}'\mathbf{V}_{(\Omega)}\mathbf{c}\right| - \frac{1}{2}(\mathbf{c}'\mathbf{y} - \mathbf{0})'\left(\mathbf{c}'\mathbf{V}_{(\Omega)}\mathbf{c}\right)^{-1}(\mathbf{c}'\mathbf{y} - \mathbf{0}) \\ &= -\left(\frac{mT - r(\mathbf{X})}{2}\right) ln(2\pi) - \frac{1}{2} ln\left|\mathbf{c}'\mathbf{V}_{(\Omega)}\mathbf{c}\right| - \frac{1}{2}\mathbf{y}'\mathbf{c}\left(\mathbf{c}'\mathbf{V}_{(\Omega)}\mathbf{c}\right)^{-1}\mathbf{c}'\mathbf{y}. \end{aligned} \tag{29}$$

Notice that $-\left(\frac{mT - r(\mathbf{X})}{2}\right) ln(2\pi)$ is a constant that does not depend on the unknown variance component in the model, and therefore, can be ignored. So that equation (29) become

$$\begin{aligned} ln\, L\left(\mathbf{V}_{(\Omega)}\right) &= -\frac{1}{2}\left(ln\left|\mathbf{c}'\mathbf{V}_{(\Omega)}\mathbf{c}\right|\right) - \frac{1}{2}\left(\mathbf{y}'\mathbf{c}\left(\mathbf{c}'\mathbf{V}_{(\Omega)}\mathbf{c}\right)^{-1}\mathbf{c}'\mathbf{y}\right) \\ &= -\frac{1}{2}\left(ln\left|\mathbf{V}_{(\Omega)}\right| + ln\left|\mathbf{X}'\mathbf{V}_{(\Omega)}\mathbf{X}\right|\right) - \frac{1}{2}(\mathbf{y}'\mathbf{P}\mathbf{y}) \end{aligned} \tag{30}$$

with $\mathbf{P} = \mathbf{V}_{(\Omega)}^{-1} - \mathbf{V}_{(\Omega)}^{-1}\mathbf{X}\left(\mathbf{X}'\mathbf{V}_{(\Omega)}^{-1}\mathbf{X}\right)^{-1}\left(\mathbf{X}'\mathbf{V}_{(\Omega)}^{-1}\right)$.

The partial derivative of the $\ln L\left(\mathbf{V}_{(\Omega)}\right)$ at equation (30) is

$$\begin{aligned} \frac{\partial\, ln\, L\left(\mathbf{V}_{(\Omega)}\right)}{\partial\Omega_j} &= -\frac{1}{2}\frac{\partial\left(ln\left|\mathbf{V}_{(\Omega)}\right|\right)}{\partial\Omega_j} - \frac{1}{2}\frac{\partial\left(ln\left|\mathbf{X}'\mathbf{V}_{(\Omega)}\mathbf{X}\right|\right)}{\partial\Omega_j} - \frac{1}{2}\frac{\partial(\mathbf{y}'\mathbf{P}\mathbf{y})}{\partial\Omega_j} \\ &= -\frac{1}{2} tr\left(\mathbf{V}_{(\Omega)}^{-1}\mathbf{V}_{(\Omega_j)}\right) + \frac{1}{2} tr\left(\left(\mathbf{X}'\mathbf{V}_{(\Omega)}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{V}_{(\Omega)}^{-1}\mathbf{V}_{(\Omega_j)}\mathbf{V}_{(\Omega)}^{-1}\mathbf{X}\right) + \frac{1}{2}\mathbf{y}'\mathbf{P}\mathbf{V}_{(\Omega_j)}\mathbf{P}\mathbf{y} \\ &= -\frac{1}{2} tr\left(\left(\mathbf{V}_{(\Omega)}^{-1} - \mathbf{V}_{(\Omega)}^{-1}\mathbf{X}\left(\mathbf{X}'\mathbf{V}_{(\Omega)}^{-1}\mathbf{X}\right)^{-1}\left(\mathbf{X}'\mathbf{V}_{(\Omega)}^{-1}\right)\right)\mathbf{V}_{(\Omega_j)}\right) + \frac{1}{2}\mathbf{y}'\mathbf{P}\mathbf{V}_{(\Omega_j)}\mathbf{P}\mathbf{y} \end{aligned}$$

$$= -\frac{1}{2} tr\left(\mathbf{PV}_{(\Omega_j)}\right) + \frac{1}{2}\mathbf{y}'\mathbf{PV}_{(\Omega_j)}\mathbf{Py}. \tag{31}$$

Resolve the equation (31) by $\frac{\partial \ln L(\mathbf{V}_{(\Omega)})}{\partial \Omega_j} = 0$, so that

$$\tfrac{1}{2} tr\left(\mathbf{PV}_{(\Omega_j)}\right) = \tfrac{1}{2}\mathbf{y}'\mathbf{PV}_{(\Omega_j)}\mathbf{Py}. \tag{32}$$

The equation (32) is solved simultaneously and often iteratively. The REML estimator of Ω is obtained iteratively from (27) by replacing the ML quantities by the REML. The (*j*, *k*)-th element of REML Information matrix is

$$\begin{aligned}\left(\boldsymbol{\Psi}_{REML(\Omega)}\right)_{jk} &= -E\left(\frac{\partial^2 \ln L\left(\mathbf{V}_{(\Omega)}\right)}{\partial\Omega_j \partial\Omega_k}\right)\\ &= \frac{1}{2} tr\left(\mathbf{PV}_{(\Omega_j)}\mathbf{PV}_{(\Omega_k)}\right).\end{aligned}$$

At the convergence of the iterations, will be get the REML estimator $\widehat{\Omega}_{REML}$ of Ω. Asymptotically, the covariance matrix, $\overline{\mathbf{V}}(\widehat{\Omega}_{REML}) \approx \overline{\mathbf{V}}(\widehat{\Omega}_{ML})$ and $\overline{\mathbf{V}}(\widehat{\boldsymbol{\beta}}_{REML}) \approx \overline{\mathbf{V}}(\widehat{\boldsymbol{\beta}}_{ML})$, provided *p* is fixed.

The MSE Estimator of the EBLUP

Because the Rao-Yu model and the dynamic model are a special case of the general linear mixed model that cover many small area models and the covariance matrices have block diagonal form when arranged by area, so that assuming that the variance components, Ω, are known, the Mean Square Error (MSE) of the BLUP is,

$$MSE\left(\hat{y}_{iT_{BLUP}}\right) = g_{1iT}(\Omega) + g_{2iT}(\Omega) \tag{33}$$

with

$$g_{1iT}(\Omega) = \mathbf{k}_i'(\mathbf{G}_i - \mathbf{G}_i\mathbf{Z}_i'\mathbf{V}_i^{-1}\mathbf{Z}\mathbf{G}_i)\mathbf{k}_i, \tag{34}$$

and

$$g_{2iT}(\Omega) = \mathbf{d}_i'\left(\sum_i \mathbf{X}_i'\mathbf{V}_i^{-1}\mathbf{X}_i\right)^{-1}\mathbf{d}_i, \tag{35}$$

where $\mathbf{d}_i' = \mathbf{x}_{iT}' - \mathbf{b}_i'\mathbf{X}_i'$, with $\mathbf{b}_i' = \mathbf{k}_i'\mathbf{G}_i\mathbf{Z}_i'\mathbf{V}_i^{-1}$,

To obtained the MSE of the BLUP under the Rao-Yu model, replace $\mathbf{k}_i = (1, \mathbf{0}_{T-1}', 1)'$, $\mathbf{G}_{\boldsymbol{i}} = \begin{pmatrix}\sigma_v^2 & \mathbf{o}_T' \\ \mathbf{o}_T & \sigma^2\boldsymbol{\Gamma}\end{pmatrix}$, $\mathbf{Z}_i = (\mathbf{1}_T, \mathbf{I}_T)$ and $\mathbf{V}_{\boldsymbol{i}} = (\mathbf{J}_T + \sigma^2\boldsymbol{\Gamma} + \boldsymbol{\Sigma}_i)$, thus (34) become

$$\begin{aligned} g_{1iT_{RY}}(\Omega) &= (1, \mathbf{0}_{T-1}', 1)\left[\begin{pmatrix}\sigma_v^2 & \mathbf{o}_T' \\ \mathbf{o}_T & \sigma^2\boldsymbol{\Gamma}\end{pmatrix} - \begin{pmatrix}\sigma_v^2\mathbf{1}_T' \\ \sigma^2\boldsymbol{\Gamma}\end{pmatrix}\mathbf{V}_i^{-1}(\sigma_v^2\mathbf{1}_T, \sigma^2\boldsymbol{\Gamma})\right]\begin{pmatrix}1 \\ \mathbf{0}_{T-1} \\ 1\end{pmatrix}\\ &= (1, \mathbf{0}_{T-1}', 1)\left[\begin{pmatrix}\sigma_v^2 & \mathbf{o}_T' \\ \mathbf{o}_T & \sigma^2\boldsymbol{\Gamma}\end{pmatrix} - \begin{pmatrix}\sigma_v^2\mathbf{1}_T' \\ \sigma^2\boldsymbol{\Gamma}\end{pmatrix}\mathbf{V}_i^{-1}(\sigma_v^2\mathbf{1}_T, \sigma^2\boldsymbol{\Gamma})\right]\begin{pmatrix}1 \\ \mathbf{0}_{T-1} \\ 1\end{pmatrix}\\ &= \left[(1, \mathbf{0}_{T-1}', 1)\begin{pmatrix}\sigma_v^2 & \mathbf{o}_T' \\ \mathbf{o}_T & \sigma^2\boldsymbol{\Gamma}\end{pmatrix} - (1, \mathbf{0}_{T-1}', 1)\begin{pmatrix}\sigma_v^2\mathbf{1}_T' \\ \sigma^2\boldsymbol{\Gamma}\end{pmatrix}\mathbf{V}_i^{-1}(\sigma_v^2\mathbf{1}_T, \sigma^2\boldsymbol{\Gamma})\right]\begin{pmatrix}1 \\ \mathbf{0}_{T-1} \\ 1\end{pmatrix}\end{aligned}$$

$$= (\sigma_v^2, \sigma^2\boldsymbol{\gamma}_T')\begin{pmatrix}1\\ \mathbf{0}_{T-1}\\ 1\end{pmatrix} - (\sigma_v^2\mathbf{1}_T' + \sigma^2\boldsymbol{\gamma}_T')\mathbf{V}_i^{-1}(\sigma_v^2\mathbf{1}_T, \sigma^2\boldsymbol{\Gamma})\begin{pmatrix}1\\ \mathbf{0}_{T-1}\\ 1\end{pmatrix}$$
$$= (\sigma_v^2 + \sigma^2\gamma_{TT}) - (\sigma_v^2\mathbf{1}_T' + \sigma^2\boldsymbol{\gamma}_T')\mathbf{V}_i^{-1}(\sigma_v^2\mathbf{1}_T + \sigma^2\boldsymbol{\gamma}_T)$$
$$= (\sigma_v^2 + \sigma^2(1-\rho^2)^{-1}) - (\sigma_v^2\mathbf{1}_T' + \sigma^2\boldsymbol{\gamma}_T')(\mathbf{J}_T + \sigma^2\boldsymbol{\Gamma} + \boldsymbol{\Sigma}_i)^{-1}(\sigma_v^2\mathbf{1}_T + \sigma^2\boldsymbol{\gamma}_T) \qquad (36)$$

and (35) become

$$g_{2iT_{RY}}(\Omega) = \mathbf{x}_{iT}'\left(\sum_i \mathbf{X}_i'\mathbf{V}_i^{-1}\mathbf{X}_i\right)^{-1}\mathbf{x}_{iT}$$
$$+(\sigma_v^2\mathbf{1}_T + \sigma^2\boldsymbol{\gamma}_T)'\mathbf{V}_i^{-1}\mathbf{X}_i'\left(\sum_i \mathbf{X}_i'\mathbf{V}_i^{-1}\mathbf{X}_i\right)^{-1}\mathbf{X}_i\mathbf{V}_i^{-1}(\sigma_v^2\mathbf{1}_T + \sigma^2\boldsymbol{\gamma}_T)$$
$$-2(\sigma_v^2\mathbf{1}_T + \sigma^2\boldsymbol{\gamma}_T)'\mathbf{V}_i^{-1}\mathbf{X}_i'\left(\sum_i \mathbf{X}_i'\mathbf{V}_i^{-1}\mathbf{X}_i\right)^{-1}\mathbf{x}_{iT}. \qquad (37)$$

To obtained the MSE of the BLUP under the dynamic model, replace $\mathbf{k}_i = \mathbf{k}_i^* = (\rho^{T-1}, \mathbf{0}_{T-1}', 1)'$, $\mathbf{G}_i = \mathbf{G}_i^* = \begin{pmatrix}\sigma_{v^*}^2 & \mathbf{o}_T'\\ \mathbf{o}_T & \sigma^2\boldsymbol{\Gamma}_{u^*}\end{pmatrix}$, $\mathbf{Z}_i = \mathbf{Z}_i^* = (\boldsymbol{\rho}_T, \mathbf{I}_T)$ and $\mathbf{V}_i = \mathbf{V}_i^* = (\sigma_{v^*}^2\boldsymbol{\Gamma}_{v^*} + \sigma^2\boldsymbol{\Gamma}_{u^*} + \boldsymbol{\Sigma}_i)$, thus (34) become

$$g_{1iT_{DYN}}(\Omega) = (\rho^{T-1}, \mathbf{0}_{T-1}', 1)\left[\begin{pmatrix}\sigma_{v^*}^2 & \mathbf{o}_T'\\ \mathbf{o}_T & \sigma^2\boldsymbol{\Gamma}_{u^*}\end{pmatrix} - \begin{pmatrix}\sigma_{v^*}^2\boldsymbol{\rho}_T'\\ \sigma^2\boldsymbol{\Gamma}_{u^*}\end{pmatrix}\mathbf{V}_i^{*^{-1}}(\sigma_{v^*}^2\boldsymbol{\rho}_T, \sigma^2\boldsymbol{\Gamma}_{u^*})\right]\begin{pmatrix}\rho^{T-1}\\ \mathbf{0}_{T-1}\\ 1\end{pmatrix}$$
$$= (\sigma_{v^*}^2\rho^{T-1}, \sigma^2\boldsymbol{\gamma}_{u^*,T}')\begin{pmatrix}\rho^{T-1}\\ \mathbf{0}_{T-1}\\ 1\end{pmatrix} - (\rho^{T-1}\sigma_{v^*}^2\boldsymbol{\rho}_T' + \sigma^2\boldsymbol{\gamma}_{u^*,T}')\mathbf{V}_i^{*^{-1}}(\sigma_{v^*}^2\boldsymbol{\rho}_T, \sigma^2\boldsymbol{\Gamma}_{u^*})\begin{pmatrix}\rho^{T-1}\\ \mathbf{0}_{T-1}\\ 1\end{pmatrix}$$
$$= \left(\sigma_{v^*}^2\rho^{2(T-1)} + \sigma^2\sum_{t=1}^{T-1}\rho^{2(t-1)}\right) - (\sigma_{v^*}^2\boldsymbol{\gamma}_{v^*,T}' + \sigma^2\boldsymbol{\gamma}_{u^*,T}')\mathbf{V}_i^{*^{-1}}(\sigma_{v^*}^2\rho^{T-1}\boldsymbol{\rho}_T + \sigma^2\boldsymbol{\gamma}_{u^*,T}) \qquad (38)$$

and (35) become

$$g_{2iT_{DYN}}(\Omega) = \mathbf{x}_{iT}'\left(\sum_i \mathbf{X}_i'\mathbf{V}_i^{*^{-1}}\mathbf{X}_i\right)^{-1}\mathbf{x}_{iT}$$
$$+(\sigma_{v^*}^2\boldsymbol{\gamma}_{v^*,T}' + \sigma^2\boldsymbol{\gamma}_{u^*,T}')\mathbf{V}_i^{*^{-1}}\mathbf{X}_i'\left(\sum_i \mathbf{X}_i'\mathbf{V}_i^{*^{-1}}\mathbf{X}_i\right)^{-1}\mathbf{X}_i\mathbf{V}_i^{*^{-1}}(\sigma_{v^*}^2\boldsymbol{\gamma}_{v^*,T} + \sigma^2\boldsymbol{\gamma}_{u^*,T})$$
$$-2(\sigma_{v^*}^2\boldsymbol{\gamma}_{v^*,T}' + \sigma^2\boldsymbol{\gamma}_{u^*,T}')\mathbf{V}_i^{*^{-1}}\mathbf{X}_i'\left(\sum_i \mathbf{X}_i'\mathbf{V}_i^{*^{-1}}\mathbf{X}_i\right)^{-1}\mathbf{x}_{iT}. \qquad (39)$$

The estimator of MSE depends on the method used to estimate the parameters. Kackar and Harvill [11] show that

$$MSE(\hat{y}_{iT_{EBLUP}}) = MSE(\hat{y}_{iT_{BLUP}}) + E(\hat{y}_{iT_{EBLUP}} - \hat{y}_{iT_{BLUP}})^2 \qquad (40)$$

Prasad and Rao [10] proposed that

$$E(\hat{y}_{iT_{EBLUP}} - \hat{y}_{iT_{BLUP}})^2 \approx g_{3iT}(\Omega) = tr\left((\overline{\mathbf{V}}\mathbf{b})\mathbf{V}_i(\overline{\mathbf{V}}\mathbf{b})'\overline{\mathbf{V}}(\widehat{\Omega})\right), \qquad (41)$$

where $\overline{\nabla}\mathbf{b} = \text{col}_{1\le j\le 3}\left(\frac{\partial \mathbf{b}_i'}{\partial \Omega_j}\right)$. Based on (40) and (41), the EBLUP has a second order approximation to the MSE equal to,

$$MSE\left(\hat{y}_{iT_{EBLUP}}\right) \approx g_{1iT}(\Omega) + g_{2iT}(\Omega) + g_{3iT}(\Omega) \tag{42}$$

To get an actual estimator of the approximation (42), replace the paramater vector Ω by its estimator $\widehat{\Omega}$. However, $g_{1iT}(\widehat{\Omega})$ is a biased estimator for $g_{1iT}(\Omega)$, then Prasad and Rao [10] proposed a correct estimator of $MSE\left(\hat{y}_{iT_{EBLUP}}\right)$ based on REML method, that is

$$\widehat{MSE}_{REML}\left(\hat{y}_{iT_{EBLUP}}\right) = g_{1iT}\left(\widehat{\Omega}_{REML}\right) + g_{2iT}\left(\widehat{\Omega}_{REML}\right) + 2g_{3iT}\left(\widehat{\Omega}_{REML}\right). \tag{43}$$

When the estimation of parameter based on ML method, then the estimator of the $MSE\left(\hat{y}_{iT_{EBLUP}}\right)$ is equal to

$$\begin{aligned}\widehat{MSE}_{ML}\left(\hat{y}_{iT_{EBLUP}}\right) = {} & g_{1iT}\left(\widehat{\Omega}_{ML}\right) + g_{2iT}\left(\widehat{\Omega}_{ML}\right) + 2g_{3iT}\left(\widehat{\Omega}_{ML}\right) \\ & -\mathbf{b}_{\widehat{\Omega}}'\left(\widehat{\Omega}_{ML}\right)\overline{\nabla}g_{1iT}\left(\widehat{\Omega}_{ML}\right) - \mathbf{b}_{\widehat{\Omega}}'\left(\widehat{\Omega}_{ML}\right)\overline{\nabla}g_{1iT}\left(\widehat{\Omega}_{ML}\right),\end{aligned} \tag{44}$$

where

$$\overline{\nabla}g_{1iT}\left(\widehat{\Omega}_{ML}\right) = \text{col}_{1\le j\le 3}\left(\frac{\partial g_{1iT}\left(\widehat{\Omega}_{ML}\right)}{\partial \Omega_j}\right) \quad \text{and} \quad \mathbf{b}_{\widehat{\Omega}}'\left(\widehat{\Omega}_{ML}\right) = \frac{1}{2m}\left(\left(\mathbf{\Psi}_{\beta}\left(\widehat{\Omega}_{ML}\right)\right)^{-\mathbf{1}}\text{col}_{1\le j\le 3}\text{tr}\left(\left(\mathbf{\Psi}_{\beta}\left(\widehat{\Omega}_{ML}\right)\right)^{-\mathbf{1}}\frac{\partial \mathbf{\Psi}_{\beta}\left(\widehat{\Omega}_{ML}\right)}{\partial \Omega_j}\right)\right),$$

$\mathbf{\Psi}_{\beta}$ is the information matrix associated with the estimation of $\boldsymbol{\beta}$.

RESULTS

There were four models used to estimate the unemployment rate. All of them assumed that the residuals were normally distributed, except the residuals of the Fay-Herriot model did not distribute randomly and normally, so the normality assumption of the residuals was not occurred. Therefore, the model was ignored in the comparison between goodness of fit among four models. Based on the normality assumption of the residuals, two way random effects model was better than the two other models. However, the two ways random effects model assumed a random time effects is independent over time. The model was used only as a comparison to show that assumed the random time effect was independent over time was not right to estimate parameter which based on rotating panel design. It has been explained by FIGURE 2 (b).

TABLE 1. Summary of the Rao-Yu model and the dynamic model

Model	Goodness of fit (REML)	Heterogeneity over area	Heterogeneity over time-area	Autoregressive Coefficient
Rao-Yu	-759.4749	$\hat{\sigma}_v^2 = 0.000037$	$\hat{\sigma}_u^2 = 0.793629$	0.924863
Dynamic	-759.4444	$\hat{\sigma}_{v^*}^2 = 6.024855$	$\hat{\sigma}_{u^*}^2 = 0.781616$	0.919872

Table 1 is summary of the output of both the Rao-Yu model and the dynamic model. The goodness of fit of both model is almost similar.There is an interesting from the summary of their models, that is all of the estimators of components of covariance has almost similar, except estimators of variance of the random area effect. The variance of the random area effect from the dynamic model is much higher than the Rao-Yu model. The dynamic model more capable to capture the heterogeneity over area.

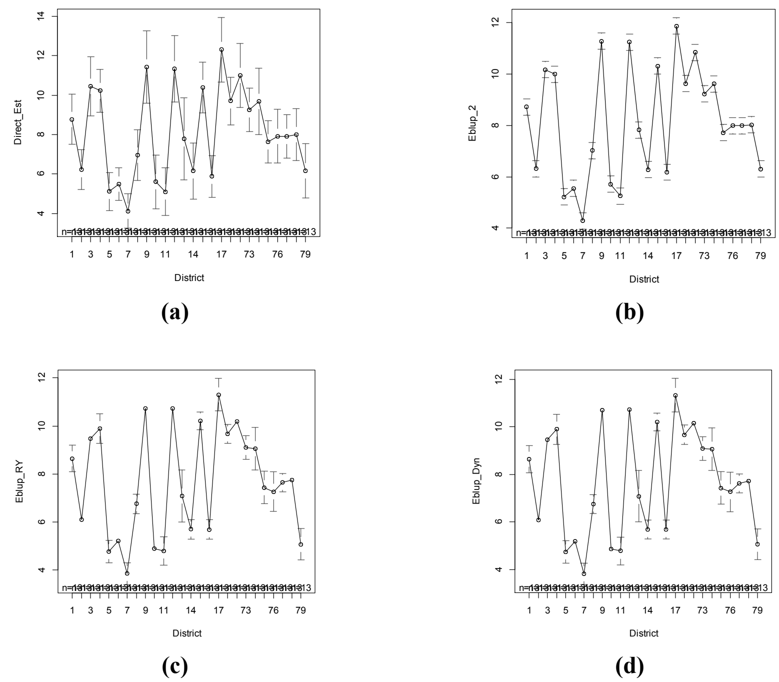


FIGURE 2. Heterogeneity of unemployment rate over area: (a) Direct estimation, (b) EBLUP two way random effect, (c) EBLUP Rao-Yu, (d) EBLUP dynamic

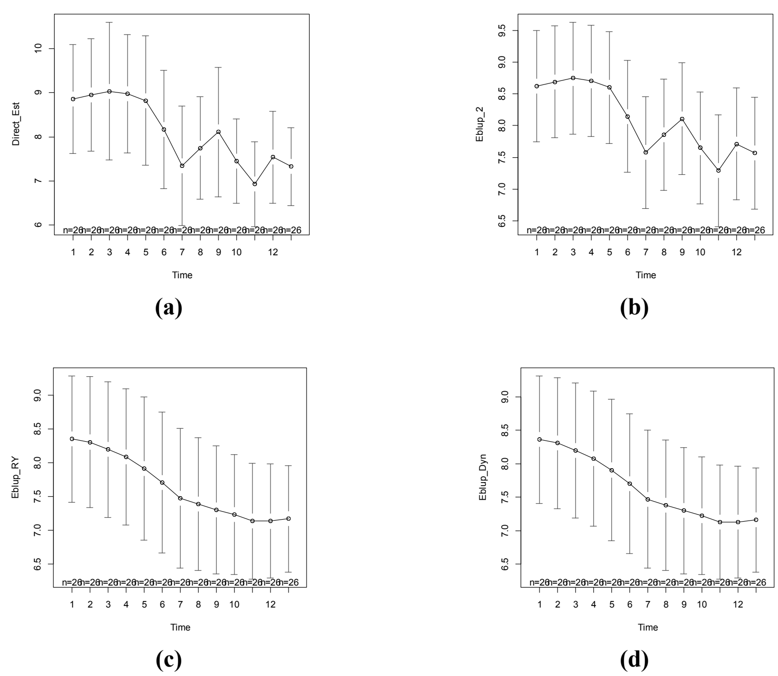


FIGURE 3. Heterogeneity over time of unemployment rate: (a) Direct estimation, (b) EBLUP Two way random effects, (c) EBLUP Rao-Yu, (d) EBLUP dynamic

FIGURE 2 and FIGURE 3 showed the heterogeneity of unemployment rate. FIGURE 2 (a) showed that the heterogeneity over area of unemployment rate was very high, but the Rao-Yu model was not able to capture it. Although there was a difference at capturing the heterogeneity over area, but their models produced EBLUP and MSE were almost similar. FIGURE 3 showed that the heterogeneity of unemployment rate over time was smaller than the heterogeneity over area. The patterns of heterogeneity over time of EBLUP that estimated by the two ways random effect model was still the same as the pattern of heterogeneity over time of the direct estimator. The Rao-Yu model and the dynamic model have smoothed the pattern of heterogeneity over time. Their model produced very large first order autocorrelation coefficients. It explained that in the context of estimation for the data designed by panel rotation, the correlation over time could not be ignored.

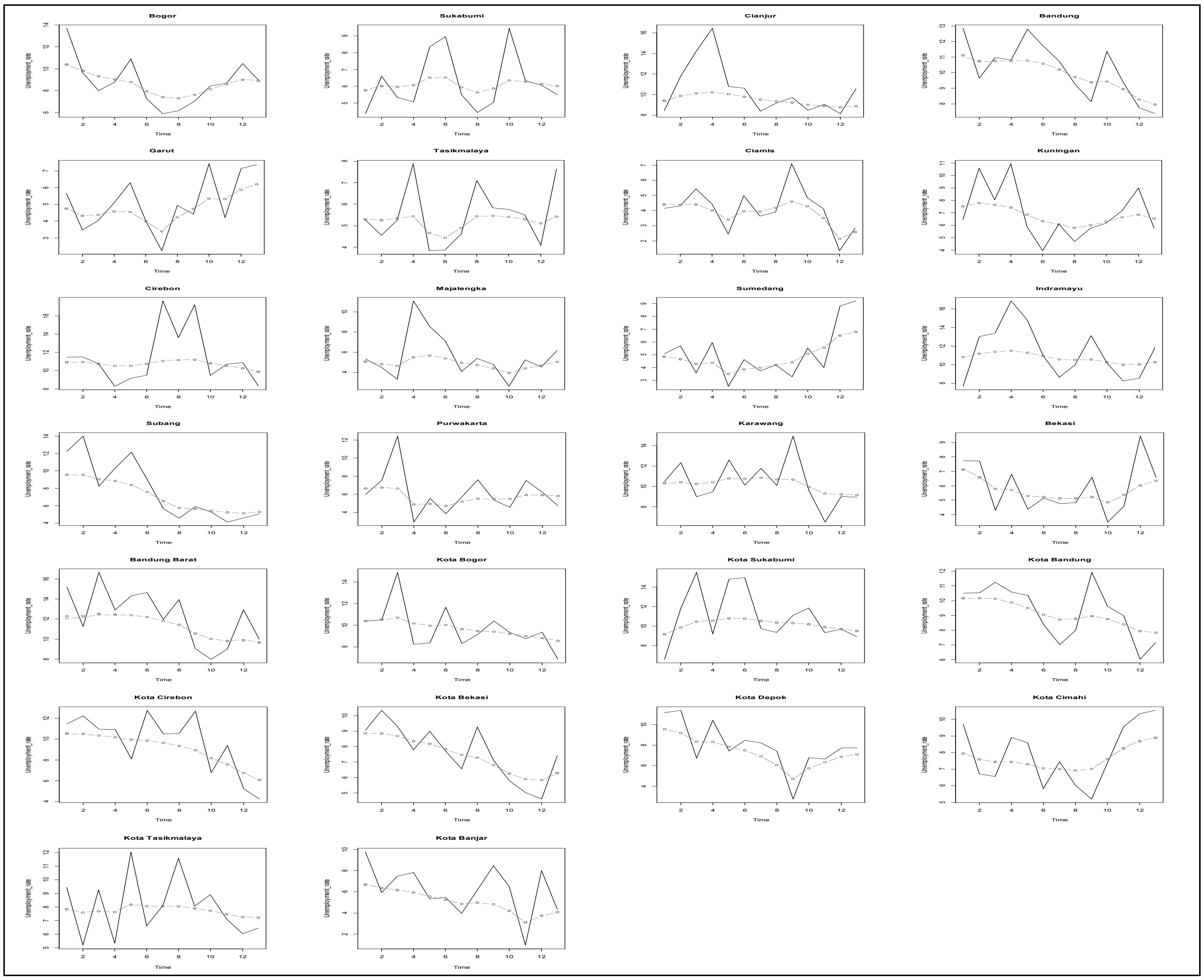


FIGURE 4. Comparison of estimator by district: Direct estimator (—), EBLUP Rao-Yu (–○–)and EBLUP Dynamic (○)

FIGURE 4 showed the comparison of estimator by district and FIGURE 5 showed the comparison of MSE. Direct estimator greatly fluctuated over time. The EBLUP from Rao-Yu model was almost similar with dynamic model so that it was difficult to distinguish because they were coincided. Their model has reduced heterogeneity both over area and over time. FIGURE 5 showed that MSE from the direct estimator was not stable but higher than both the Rao-Yu model and dynamic model. It explained that the EBLUP from both model were better than the direct estimator.

FIGURE 6 showed the estimation of unemployment rate by district at t = 13. Blue indicated the estimation was higher than the mean of all district and red indicated the estimation was lower than the mean of all district. in Bogor Municipality (71), Bandung Municipality (73) and Tasikmalaya Municipality (78), the estimation based on a direct estimation was higher than the mean, but based on estimation models Rao-Yu and dynamic models turned out to be lower. Instead, in Garut (05), Tasikmalaya (06), Bekasi Municipality (75) and Depok Municipality (76), the estimation based on the direct estimation is lower than the mean, but based on estimation models Rao-Yu and dynamic models turned out to be higher.

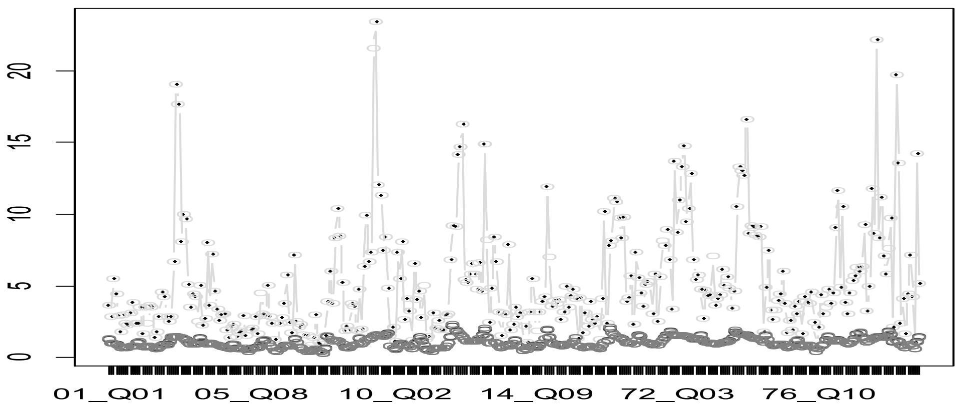
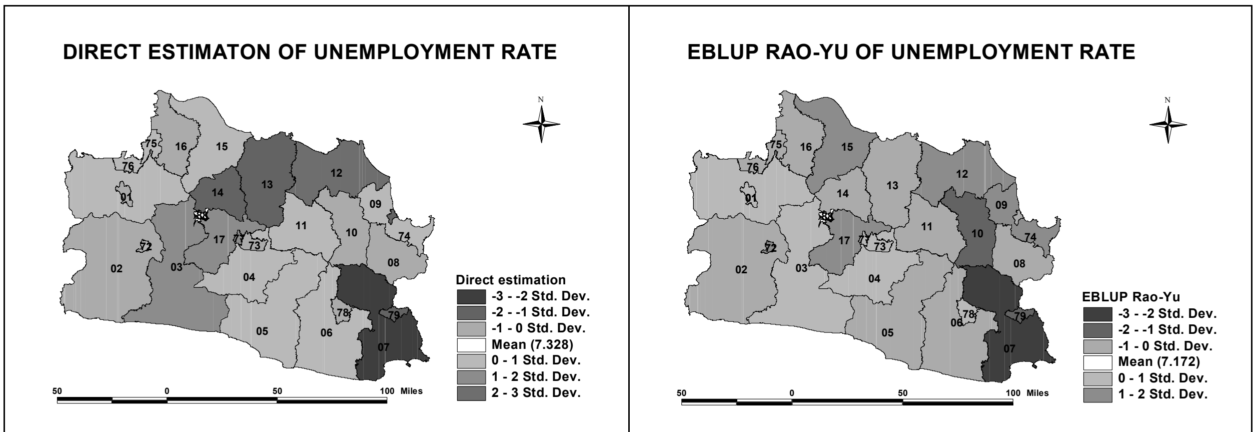
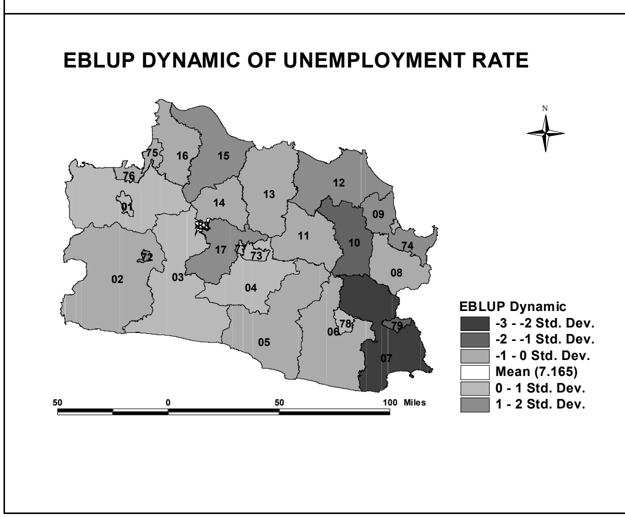


FIGURE 5. Comparison of MSE: Direct estimation (–○–), EBLUP Rao-Yu (–○–) and EBLUP Dynamic (–○–)





Code of districts:

01. Bogor	14. Purwakarta
02. Sukabumi	15. Karawang
03. Cianjur	16. Bekasi
04. Bandung	17. Bandung Barat
05. Garut	71. Bogor Municipality
06. Tasikmalaya	72. Sukabumi Municipality
07. Ciamis	73. Bandung Municipality
08. Kuningan	74. Cirebon Municipality
09. Cirebon	75. Bekasi Municipality
10. Majalengka	76. Depok Municipality
11. Sumedang	77. Cimahi Municipality
12. Indramayu	78. Tasikmalaya Municipality
13. Subang	79. Banjar Municipality

FIGURE 6.Comparison of unemployment rate from three method of estimation, at t=13.

DISCUSSION

The dynamic model was a minor modification of the Rao-Yu model by removing the stationarity requirement and modifying the random effect terms. The Rao-Yu model assumed that the random area effect was the same for every time, but the dynamic model assumed that the random area effect has changed over time according to the value of autoregressive coefficient. The dynamic model did not force the absolute value of autoregressive coefficient which was less than one. When the value of autoregressive coefficient more than one, the model corresponded to a divergent situation in which areas became progressively more disparate, but instead, when the disparity among area dissipated over time, the dynamic model was more appropriate.

In case of Sakernas, at early time the disparity over area was high, then it reduced over time. The Rao-Yu model was not able to capture this situation even it seemed there was not a disparity among areas. Otherwise, the dynamic model was very capable to capture the situation. However, both models produced the estimator was similar and better than direct estimator. Sakernas was designed by rotating panel, but both models have not been facilitating the rotation effect term, so that it was necessary to develop a model by expanding both or one of models by adding a rotation effect term. This is a challenge in the future research. Based on the ability to capture changes in the disparity among area over time, the dynamic model seems more promising to be expanded, because the disparity over area reduced over time.

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